Fuzzy Weak Bi-ideals in Near-Subtraction Semigroups

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Abstract:

In this paper, we introduce the new concept of fuzzy weak bi-ideals of near-subtraction semigroups. Some of its properties with examples were also given.

Key words:

Ideal, bi-ideal, weak bi-ideal, fuzzy ideal, fuzzy bi-ideal.

1.Introduction:

A system of the form (ϕ;o;\) where ϕ is a set of functions closed under the composition 'o' of function(hence (ϕ;o) is a function semigroup) and \ is the set theoretic subtraction is called a subtraction algebra in the sense of [1].scheine [11] showed that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions.Zelinka[13] studied a special type of subtraction algebra called atomic subtraction algebra. The study of ideals in subtraction algebra was initiated by Jun et al. who also established some basic properties. Dheena et al.[4,5] discussed and derived some properties of near-subtraction semigroup.The concept of fuzzy set was first initiated by Zadeh[12]. Narayanan et al.[9] defined the concept of generalized fuzzy ideals of near rings. Mahalakshmi et al.[7] studied the notion of bi-ideals of near-subtraction semigroups. Manikandan[8] studied fuzzy bi-ideals of near-ring and established some of their properties. Chinnadurai et al.[2] defined the concept of fuzzy bi-ideals of near-subtraction semigroups. Chinnadurai et al.[3] studied the notion of fuzzy weak bi-ideals of near- rings and established some of its properties. Motivated by this concept, we introduced fuzzy weak bi-ideals in near-subtraction semigroups and some of its properties.

2. Preliminaries:

Definition: 2.1

A non-empty set X together with binary operation "–" is said to be a ***subtraction algebra*** if it satisfies the following:

i. $x-(y-x)=x.$
ii. $x-(x-y)=y-(y-x).$
iii. $(x-y)-z=(x-z)-y$, for every x, y, z ∈ X.

Definition: 2.2

A non-empty set X together with two binary operations "–" and "•" is said to be a right ***near-subtraction semigroup*** if it satisfies the following:

i. $(X,-)$ is a subtraction algebra.
ii. $(X,\bullet)$ is a semigroup.
iii. $(x-y)z=xz-yz$, for every x, y, z ∈ X.

It is clear that $0x=0$, for all $x\in X$. Similarly, we can define a left near-subtraction semigroup. Here after a near-subtraction semigroup means only a right near-subtraction semigroup, unless mentioned otherwise.

Definition: 2.3

A near-subtraction semigroup X is said to be ***zero-symmetric*** if $x0=0$ for every $x\in X$.

Definition: 2.4

A non-empty subset S of a subtraction algebra X is said to be a ***subalgebra*** of X, if $x - y \in S$, for every $x, y \in S$.

Definition: 2.5

Let (X, –, •) be a near-subtraction semigroup.A non-empty subset I of X is called

i. A ***left ideal*** if I is a subalgebra of (X,–) and $xi - x(y - i) \in I$, for every $x, y \in X$.
ii. A ***right ideal*** if I is a subalgebra of (X,–) and $IX \subseteq I$.
iii. An ***ideal*** if I is both a left and right ideal.

Definition: 2.6

Let A and B be two subsets of X. Then the ***product*** and $*$ ***product*** defined by,
$AB = \{ab /\ a \in A, b \in B\}$ and $A * B = \{ab - a(a' - b)/\ a, a' \in A, b \in B\}$.

Definition: 2.7

An subalgebra B of X is said to be ***bi-ideal*** if $BXB \cap BX * B \subseteq B$. In case of zero-symmetric, $BXB \subseteq B$.

Definition: 2.8

A subalgebra B of X is said to be a ***weak bi-ideal*** of X if $BBB \subseteq B$.

Definition:2.9

A function μ is a mapping from X into the interval [0,1] is called a ***fuzzy set .***

Definition: 2.10

A fuzzy subset μ of X is said to be a ***fuzzy subalgebra*** of X if $x, y \in X$ implies $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

Definition: 2.10

A fuzzy subset μ is called a ***fuzzy ideal*** of x if it satisfies the following conditions:

i. $\mu(x - y) = \min\{\mu)(x), \mu(y)\}$.
ii. $\mu(y + x - y) \geq \mu(x)$.
iii. $\mu(xy) \geq \mu(y)$,
iv. $\mu\big((x + z)y - xy\big) \geq \mu(z)$, for every $x, y, z \in X$.

A fuzzy subset with (i) and (ii)is called a fuzzy left ideal of X, whereas a fuzzy subset with (i), (ii) and (iv) is called a fuzzy right ideal of X.

Definition: 2.10

Let μ and λ be any two fuzzy subsets of X. Then $\mu \cap \lambda, \mu \cup \lambda,\ \mu\lambda,\ \lambda\mu,\ \mu * \lambda$ are fuzzy subsets of X defined by,

$(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}$

$(\mu \cap \lambda)(x) = \max\{\mu(x), \lambda(x)\}$

$$(\mu - \lambda)(x) = \begin{cases} \sup_{x=y-z} \min\{\mu(y), \lambda(z)\} & if\ x\ can\ be\ expressed\ as\ x = y - z \\ 0 & otherwise \end{cases}$$

.

$$\mu\lambda(x) = \begin{cases} \sup_{x=yz} min\{\mu(y), \lambda(z)\} & if\ x\ can\ be\ expressed\ as\ x = yz \\ 0 & otherwise \end{cases}$$

$$(\mu * \lambda)(x) = \begin{cases} \sup_{x=ac-a(b-c)} min\{\mu(a), \lambda(c)\} & if\ x\ can\ be\ expressed\ as\ x = ac \\ 0 & otherwise \end{cases}$$

Definition: 2.11

A family of fuzzy set $\{\mu_i / i \in \Omega\}$ is a near-subtraction semigroup X, the intersection of $\bigcap_{i\epsilon\Omega} \mu_i$ of $\{\mu_i/i\epsilon\Omega\}$ is defined by, $\bigcap_{i\epsilon\Omega} \mu_i(x) = \inf\{\mu_i(x)/i\epsilon\Omega\}$.

Definition: 2.12

A fuzzy subalgebra μof X is said to be a ***fuzzy bi-ideal*** of X, if $(\mu X\mu) \cap (\mu X * \mu) \subseteq \mu$. In case of zero-symmetric if $\mu X\mu \subseteq \mu$.

Definition: 2.13

A fuzzy subalgebra μ of X is called a ***fuzzy X-subalgebra*** of X if,

i. μ is a fuzzy subalgebra of (X, –).
ii. $\mu(xy) \geq \mu(x)$.
iii. $\mu(xy) \geq \mu(y)$, for every $x, y, z \in X$.

A fuzzy subset with (i) and (ii) is called a fuzzy right x-subalgebra of X, whereas a fuzzy subset with (i) and (iii) is called a fuzzy left X-subalgebra of X.

Definition: 2.14

Let I be a subset of a near-subtraction semigroup X. Define a function f_I : X→ [0,1] by,

$$f_I(x) = \begin{cases} 1 & if\ x \in I \\ 0 & otherwise \end{cases}$$

For every $x \in X$. Clearly f_I is a fuzzy subset of x and is called the ***characteristic function*** of I.

3. Fuzzy weak bi-ideals in Near-subtraction Semigroups

In this section, we introduced the new concept of fuzzy weak bi-ideals of X and discuss some of its properties.

Definition: 3.1

A fuzzy subalgebra μ of X is called ***fuzzy weak bi-ideal*** of X, if $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$.

Example: 3.2

Let X = {0, a, b, c} in which "–" and "•" are defined by,

–	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

•	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then (X, –, •) is a near-subtraction semigroup. Let μ: X → [0, 1] be a fuzzy subalgebra of X defined by, μ(0)=0.9, μ(a)=0.7, μ(b)=0.6, μ(c)=0.4.Then μ is a fuzzy weak bi-ideal of X.

Theorem: 3.3

Let μ be a fuzzy subalgebra of X, then μ is a fuzzy weak bi-ideal of X iff $\mu\mu\mu \subseteq \mu$.

Proof:

Assume that μ is a fuzzy weak bi-ideal of X. Let $x, y, z, y_1, y_2 \in$ X such that $x = yz$ and $y = y_1y_2$.Then,

$$\mu\mu\mu(x) = \sup_{x=yz} \{min\{(\mu\mu)(y), \mu(z)\}\}$$
$$= \sup_{x=yz} \left\{min\left\{\sup_{y=y_1y_2} min\{\mu(y_1), \mu(y_2)\}\right\}\right\}$$

$= \sup_{x=yz} \sup_{y=y_1y_2} \{min\{min\{\mu(y_1),\mu(y_2)\},\mu(z)\}\}$

$= \sup_{x=y_1y_2z} \{min\{\mu(y_1),\mu(y_2),\mu(z)\}\}$

[Since μ is a fuzzy weak bi-deal of X, $\mu(xyz) \geq min\{\mu(x),\mu(y),\mu(z)\}$]

$\leq \sup_{x=y_1y_2z} \mu(y_1y_2z)$

If can be expressed as $= yz$, then $(\mu\mu\mu)(x) \leq \mu(x)$. In both cases $\mu\mu\mu \subseteq \mu$.

Conversely,

Assume that $\mu\mu\mu \subseteq \mu$. For $x', x, y, z \in X$. Let x' be such that $x' = xyz$.Then,

$$\begin{aligned}
\mu(xyz) &= \mu(x') \geq (\mu\mu\mu)(x') \\
&= \sup_{x'=pq}\{min\{(\mu\mu)(p),\mu(q)\}\} \\
&= \sup_{x'=pq}\{min\{\sup_{p=p_1p_2} min\{\mu(p_1),\mu(p_2)\},\mu(q)\}\} \\
&= \sup_{x=p_1p_2q} \{min\{\mu(p_1),\mu(p_2),\mu(q)\}\} \\
&\geq \min\{\mu(x),\mu(y),\mu(z)\}
\end{aligned}$$

Hence $\mu(xyz) \geq \min\{\mu(x),\mu(y),\mu(z)\}$.

Theorem:3.4

Let $\{\mu_i / i \in \Omega\}$ be family of fuzzy weak bi-ideals of a near-subtraction semigroup X, then $\bigcap_{i\in\Omega} \mu_i$ is also a fuzzy weak bi-ideal of X, where Ω is any index set.

Proof :

Let $\{\mu_i\}_{i\epsilon\Omega}$ be a family of a fuzzy weak bi-ideals of X.

Let $x, y, z \,\epsilon\, X$ and $\mu = \bigcap_{i\epsilon\Omega} \mu_i$. Then $\mu(x) = \bigcap_{i\epsilon\Omega} \mu_i(x) = \inf_{i\epsilon\Omega} \mu_i(x)$

$$\begin{aligned}
\mu(x-y) &= \inf_{i\epsilon\Omega} \mu_i(x-y) \\
&\geq \inf_{i\epsilon\Omega} \min\{\mu_i(x),\mu_i(x)\} \\
&= min\{\inf_{i\epsilon\Omega} \mu_i(x), \inf_{i\epsilon\Omega} \mu_i(y)\} \\
&= min\{\bigcap_{i\epsilon\Omega} \mu_i(x), \bigcap_{i\epsilon\Omega} \mu_{i\epsilon\Omega}(x)\} \\
&= min\{\mu(x),\mu(y)\}
\end{aligned}$$

And,
$$\begin{aligned}
\mu(xyz) &= \inf_{i\epsilon\Omega} \mu_i(xyz) \\
&\geq \inf_{i\epsilon\Omega} \min\{\mu_i(x),\mu_i(y),\mu_i(z)\} \\
&= min\{\inf_{i\epsilon\Omega} \mu_i(x), \inf_{i\epsilon\Omega} \mu_i(y), \inf_{i\epsilon\Omega} \mu_i(z)\} \\
&= \min\{\bigcap_{i\epsilon\Omega} \mu_i(x), \bigcap_{i\epsilon\Omega} \mu_i(y), \bigcap_{i\epsilon\Omega} \mu_i(z)\} \\
&= \min\{\mu(x),\mu(y),\mu(z)\}.
\end{aligned}$$

Theorem: 3.5

Let μ and λ be fuzzy weak bi-ideals of X. Then the products μλ and λμ are also fuzzy weak bi-deals of X.

Proof :

Let μ and λ be fuzzy weak bi-ideals of X. Then,

$$\begin{aligned}
(\mu\lambda)(x-y) &= \sup_{x-y=ab} \min\{\mu(a),\lambda(b)\} \\
&\geq \sup_{x-y=a_1b_1-a_2b_2<(a_1-a_2)(b_1-b_2)} \min\{\mu(a_1-a_2),\lambda(b_1-b_2)\} \\
&\geq \sup min\{\{\min\{\mu(a_1),\mu(a_2)\}, \min\{\lambda(b_1),\lambda(b_2)\}\} \\
&= \sup \min\{\{\min\{\mu(a_1),\lambda(b_1)\}, \min\{\mu(a_2),\lambda(b_2)\}\} \\
&\geq min\{\sup_{x=a_1b_1} \min\{\mu(a_1),\lambda(b_1)\}, \sup_{y=a_2b_2} \min\{\mu(a_2),\lambda(b_2)\}\} \\
&= \min\{(\mu\lambda)(x),(\mu\lambda)(y)\}
\end{aligned}$$

It follows that μλ is a fuzzy subalgebra of X. Further,

$(\mu\lambda)\,(\mu\lambda)\,(\mu\lambda) = \mu\lambda\,(\mu\lambda\mu)\,\lambda$

$\subseteq (\mu\lambda)(\lambda\lambda\lambda)\,\lambda$ (since λ is a fuzzy weak bi-ideal of X)

$\subseteq \mu\,(\lambda\lambda\lambda)$ (since λ is a fuzzy weak bi-ideal of X)

$\subseteq$ μλ

Therefore μλ is a fuzzy weak bi-ideal of X. Similarly, λμ is a fuzzy weak bi-ideal of X.

Theorem: 3.6

Every fuzzy bi-ideal of X is a fuzzy weak bi-ideal of X.

Proof:

Assume that μ is a fuzzy bi-ideal of X. Then $\mu X\mu \cap (\mu X) * \mu \subseteq \mu$. We have, $\mu\mu\mu \subseteq \mu X\mu$ and $\mu\mu\mu \subseteq (\mu X) * \mu$. This implies that $\mu\mu\mu \subseteq \mu X \cap (\mu X) * \mu \subseteq \mu$. Therefore, μ is a fuzzy bi-ideal of X.

However, the converse of the theorem 3.6 is not true in general which is demonstrated by the following example:

Example: 3.7

Let X = {0, a, b, c} with two binary operations "–" and "•" is defined as follows:

–	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

•	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Then, (X, –, •) is a near-subtraction semigroup. Let μ: X→[0,1] be fuzzy subalgebra of X defined by, μ(0) = 0.8, μ(a) = 0.5 = μ(b), μ(c) = 0.7. Clearly, μ is a fuzzy weak bi-ideal of X. But $\mu(cbc) \ngeq min\{\mu(c), \mu(c)\}$ Therefore, μ is not a fuzzy bi-ideal of X.

Theorem: 3.8

Every fuzzy ideal of X is a fuzzy weak bi-deal of X.

Proof:

By theorem 3.7 in [12], Every fuzzy ideal of X is a fuzzy bi-ideal of X. Also by theorem 3.6, we have, every fuzzy bi-ideal of X is a fuzzy weak bi-ideal of X. Therefore, μ is a fuzzy weak bi-ideal of X.

Example: 3.9

Let X = {0, a, b, c} with two binary operations "–" and "•" is defined as follows:

–	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

•	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	b	c
c	0	a	b	c

Then, (X, –, •) is a near-subtraction semigroup. Let μ: X→[0,1] be fuzzy subalgebra of X defined by, μ(0) = 0.7, μ(a) = μ(b) = 0.2, μ(c) = 0.6. Clearly, μ is a fuzzy weak bi-ideal of X. But $\mu(0c) \ngeq \mu(0)$. Therefore, μ is not a fuzzy ideal of X.

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