Instability Phenomenon Arising in Homogeneous Porous Media by Crank-Nicolson Finite Difference Method

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ABSTRACT

During secondary oil recovery process, when water is injected in oil formatted area to recover remaining oil, the important phenomenon Instability (fingering) occurs due to the injecting force, difference of viscosity and wettability of the water and oil gives rise to the protuberances (instability) at common interface. The injected water will shoot through the porous medium at relatively high speed through the inter connected capillaries and its shape is as good as fingers, therefore it is also called fingering phenomenon and fingers are unstable due to the injecting force, so it is also known as instability phenomenon. The mathematical formulation yields to a non-linear partial differential equation known as Boussinesq equation. Its solution has been obtained by using unconditionally stable Crank-Nicolson finite difference scheme with appropriate initial and boundary conditions. The solution has been compared with the solution of this phenomenon. The solution represents saturation of injected water which is increasing as length of fingers x - increases for given time t>0. The solution represents saturation of injected water occupied by average cross-sectional area of schematic fingers for instability phenomenon.

Keywords:

Instability phenomenon, immiscible displacement, Crank-Nicolson finite difference scheme, fluid flow through porous medium.

INTRODUCTION

This paper we discusses the fingering phenomenon by choosing small cylindrical piece of porous matrix from large natural oil basin with three sides are impermeable except one end. For the sake of mathematical study, for one-dimensional study of displacement process, taking vertical cross-sectional area of the cylindrical piece of porous matrix of length L which is shown as rectangle having irregular fingers in right side of common interface x =0
(figure 1), [Scheidegger and Johnson] suggested to replace irregular fingers by schematic fingers of rectangular shape which is shown by the figure 4. They considered average crosssectional area occupied by schematic fingers, as saturation of injected water which is shown by rectangle in right side of common interface x=0 (figure 4).

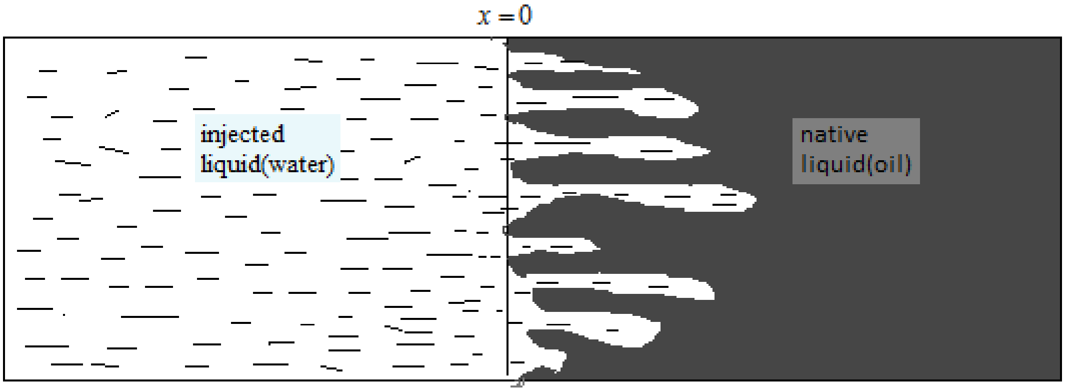


Figure 1: The formation of natural fingers during instability phenomenon in the porous medium

The present paper describes instability (fingering) phenomenon in double phase flow of two immiscible fluids (water and oil) through homogeneous porous medium. The displacement of native oil by another of lesser viscosity injected water provides a well-developed finger. Thus, when injected water flowing through porous medium and displacing native fluid oil, then instead of regular displacement of whole front, protuberances (fingers) take place which shoots through the porous medium at relatively very high speed. In the present paper our interest is to solve non-linear partial differential equation for one-dimensional instability phenomenon arising in homogeneous porous medium during secondary oil recovery process. The numerical solution of this equation has been obtained by using Crank-Nicolson scheme of finite difference method.

STATEMENT OF THE PROBLEM

To study the phenomenon of instability in homogeneous porous medium we choose a cylindrical piece of porous matrix of length L whose three sides are impermeable except one end, from where water is injected. For mathematical study we take vertical cross-sectional area of this cylindrical piece of porous matrix which is rectangle and open end will be the common interface x=0 .
Let the water be injected at x=0 then due to the injecting force and viscosity difference the
protuberances or instability may arise which is due to the displacement of oil by water injection through inter connected capillaries. The length x of the fingers is being measured in the direction of displacement. Scheidegger and Johnson [15] suggested replacing these irregular fingers by schematic fingers of rectangular size.

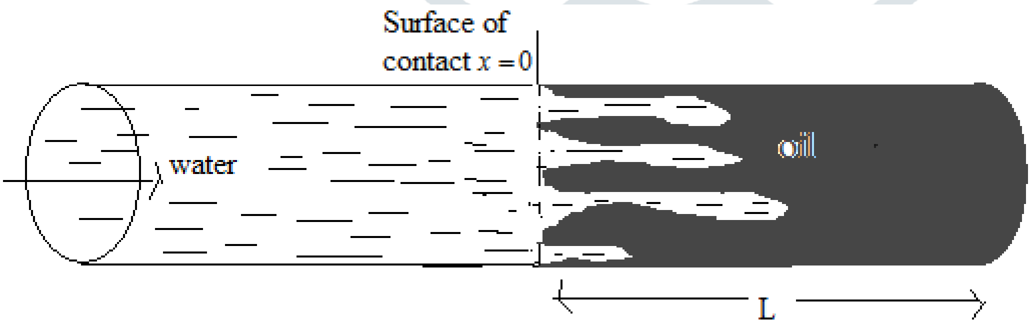


Figure 2: The formation of fingers in the cylindrical piece of porous medium
When water is injected, at common interface x=0 the oil will be displaced from the porous media. Hence, seepage velocities of injected water (V_w) and native oil (V_o) are expressed by Darcy's law for these two immiscible fluids as Bear [1].

$$V_w = -\frac{k_w}{\delta_w} K \frac{\partial P_w}{\partial x} \qquad (1.1)$$

$$V_o = -\frac{k_o}{\delta_o} K \frac{\partial P_o}{\partial x} \qquad (1.2)$$

K is the permeability of the homogeneous porous medium.

k_w and k_o are the relative permeability of water and oil, which are the functions of the saturation of the water and oil S_w and S_o respectively.

P_w and P_o are the pressure of water and oil.

δ_w and δ_o are the constant kinematics viscosities of injecting water and displacing oil respectively.

The injecting water and displaced oil also satisfy the equation of continuity for their constant phase densities as,

$$P\frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \qquad (1.3)$$

$$P\frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \qquad (1.4)$$

where P is the porosity of the porous medium.

Substituting the values of the seepage velocities V_w and V_o from the equation (1.1) and (1.2) into the equations (1.3) and (1.4) respectively. We get,

$$P\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial t}\left(\frac{k_w}{\delta_w}K\frac{\partial P_w}{\partial x}\right) \qquad (1.5)$$

$$P\frac{\partial S_o}{\partial t} = \frac{\partial}{\partial t}\left(\frac{k_o}{\delta_o}K\frac{\partial P_o}{\partial x}\right) \qquad (1.6)$$

When water is injected at common interface flow of injected water takes place only due to the capillary pressure P_c , which is defined as pressure difference between native oil and injected water [5,6,7].

$$P_c = P_o - P_w \qquad (1.7)$$

Eliminating $\frac{\partial P_w}{\partial x}$ from equation (1.5) and (1.7), we obtain

$$P\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{k_w}{\delta_w}\left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right) \qquad (1.8)$$

For more simplification adding (1.6) and (1.8) and using the relation $S_w + S_o = 1$, we get

$$\frac{\partial}{\partial x}\left(K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial P_o}{\partial x} - K\frac{k_w}{\delta_w}\frac{\partial P_c}{\partial x}\right) = 0 \qquad (1.9)$$

Now, integrating above equation w.r.t x, we get

$$\left(K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial P_o}{\partial x} - K\frac{k_w}{\delta_w}\frac{\partial P_c}{\partial x}\right) = -A(t) \qquad (1.10)$$

Where, $A(t)$ is constant of integration. Simplification of the equation (1.10) gives

$$\frac{\partial P_o}{\partial x} = \frac{-A(t)}{K\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)} + \frac{\frac{\partial P_c}{\partial x}}{1+\frac{k_o}{k_w}\frac{\delta_w}{\delta_o}} \qquad (1.11)$$

Using (1.11) in (1.8) we have,

$$P\,\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{K\frac{k_o}{\delta_o}\frac{\partial P_c}{\partial x}}{1+\frac{k_o}{k_w}\frac{\delta_w}{\delta_o}} + \frac{A(t)}{1+\frac{k_o}{k_w}\frac{\delta_w}{\delta_o}}\right] = 0 \qquad (1.12)$$

For more simplification we replace the pressure of oil P_o as

$$P_o = \frac{P_o + P_w}{2} + \frac{P_o - P_w}{2}$$
$$= \bar{P} + \frac{1}{2}P_c \qquad (1.13)$$

Where, $\bar{P}$ the mean pressure which is constant, and also using the relation (1.7) we get,

$$\frac{\partial P_o}{\partial x} = \frac{1}{2}\frac{\partial P_c}{\partial x} \qquad (1.14)$$

Using (1.14) in (1.10) and substituting the value of $A(t)$, we get

$$P\,\frac{\partial S_w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(K\,\frac{k_w}{\delta_w}\,\frac{\partial P_c}{\partial S_w}\,\frac{\partial S_w}{\partial x}\right) = 0 \qquad (1.15)$$

For the further simplification we use the following standard relations

$$K_w = S_w \;\;\&\;\; P_c = -\beta S_w$$

The equation (1.15) gives,

$$P\,\frac{\partial S_w}{\partial t} - \frac{\beta}{2}\,\frac{K}{\delta_w}\,\frac{\partial}{\partial x}\left(S_w\,\frac{\partial S_w}{\partial x}\right) = 0 \qquad (1.16)$$

The above equation (1.16) is a non-linear equation representing phenomenon of instability in homogeneous porous media.

To make the equation (1.16) dimensionless, we set the dimensionless variables as

$$X = \frac{x}{L} \text{ and } T = \frac{K\beta t}{2\delta_w L^2 P} \quad \text{where } 0 \le X \le 1,\ 0 \le T \le 1$$

The equation (1.16) reduces to

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left(S_w\,\frac{\partial S_w}{\partial X}\right) \qquad (1.17)$$

To solve this nonlinear Partial differential equation for instability phenomenon we choose the set of boundary conditions.

Let at the common interface saturation of injected water be linear function of time T then it can be expressed as

$$S_w(0,T) = T, \quad T > 0 \qquad (1.18)$$

And also let the saturation of injected water at end x=l, the end condition is

$$S_w(1,T) = 1 - T, \quad T > 0 \qquad (1.19)$$

Also, we assume that initial saturation is a linear function of X,

$$S_w(X,0) = X, \quad 0 < X \le 1 \qquad (1.20)$$

$$\frac{\partial S}{\partial T} = \frac{\partial}{\partial X}\left(S\,\frac{\partial S}{\partial X}\right)$$

$$\frac{\partial S}{\partial T} = S\left\{\frac{\partial}{\partial X}\left(\frac{\partial S}{\partial X}\right)\right\} + \left\{\frac{\partial S}{\partial X}\right\}\left\{\frac{\partial S}{\partial X}\right\}$$

$$= S\,\frac{\partial^2 S}{\partial X^2} + \left(\frac{\partial S}{\partial X}\right)^2$$

The expression (1.17) is discretized according to Crank-Nicolson Finite Difference Scheme as follows [2, 3, 4, 8]

Now, we substitute the values

$$\frac{S_i^{n+1}-S_i^{n}}{\Delta T} = S_i^{n+1/2}\left[\frac{1}{2}\left\{\frac{S_{i+1}^{n+1}-2\,S_i^{n+1}+S_{i-1}^{n+1}}{(\Delta X)^2}+\frac{S_{i+1}^{n}-2\,S_i^{n}+S_{i-1}^{n}}{(\Delta X)^2}\right\}\right]$$

$$+\left[\frac{S_{i+1}^{n+1/2}-S_{i-1}^{n+1/2}}{2(\Delta X)}\right]^2$$

$$=\left(\frac{S_i^{n}+S_i^{n+1}}{2}\right)\left[\frac{1}{2}\left\{\frac{S_{i+1}^{n+1}-2\,S_i^{n+1}+S_{i-1}^{n+1}}{(\Delta X)^2}+\frac{S_{i+1}^{n}-2\,S_i^{n}+S_{i-1}^{n}}{(\Delta X)^2}\right\}\right]$$

$$+\frac{1}{16(\Delta X)^2}\begin{bmatrix}(S_{i+1}^{n}-S_{i-1}^{n})^2+\left(S_{i+1}^{n+1}-S_{i-1}^{n+1}\right)^2\\ +2(S_{i+1}^{n}-S_{i-1}^{n})\left(S_{i+1}^{n+1}-S_{i-1}^{n+1}\right)\end{bmatrix}$$

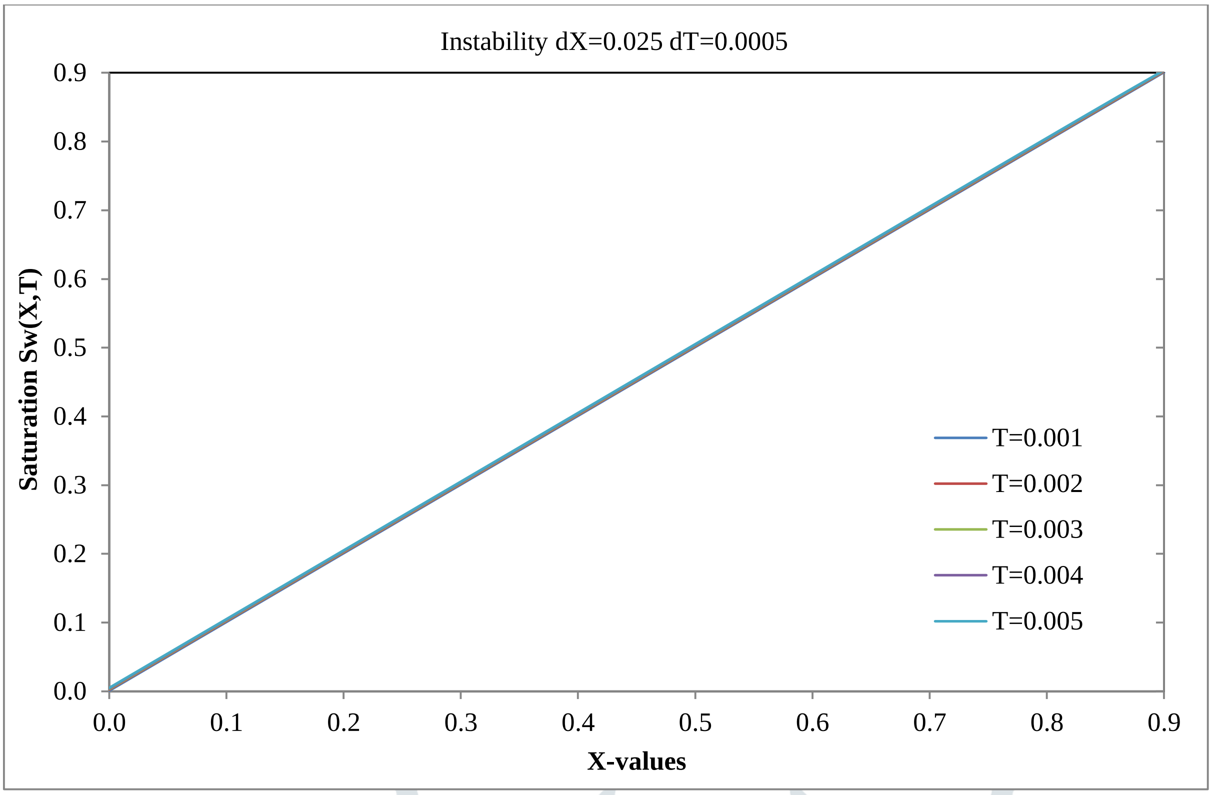
$$= C_0\left[\left[4\begin{pmatrix}(-2)\left(S_i^{n+1}\right)^2+\left(S_{i+1}^{n+1}\right)\left(S_i^{n+1}\right)+\left(S_{i-1}^{n+1}\right)\left(S_i^{n+1}\right)+\\ (S_i^{n})\left(S_{i+1}^{n+1}\right)+(S_i^{n})\left(S_{i-1}^{n+1}\right)+(-2S_i^{n}+S_{i+1}^{n}-2S_i^{n}+S_{i-1}^{n})S_i^{n+1}\\ +(S_{i+1}^{n}-2S_i^{n}+S_{i-1}^{n})S_i^{n}\end{pmatrix}\right] + \begin{bmatrix}(S_{i+1}^{n})^2+(S_{i-1}^{n})^2-2(S_{i+1}^{n})(S_{i-1}^{n})+\left(S_{i+1}^{n+1}\right)^2\\ +\left(S_{i-1}^{n+1}\right)^2-2\left(S_{i+1}^{n+1}\right)\left(S_{i-1}^{n+1}\right)\\ +2\left[(S_{i+1}^{n}-S_{i-1}^{n})\left(S_{i+1}^{n+1}\right)+(S_{i+1}^{n}-S_{i-1}^{n})\left(-S_{i-1}^{n+1}\right)\right]\end{bmatrix}\right]$$

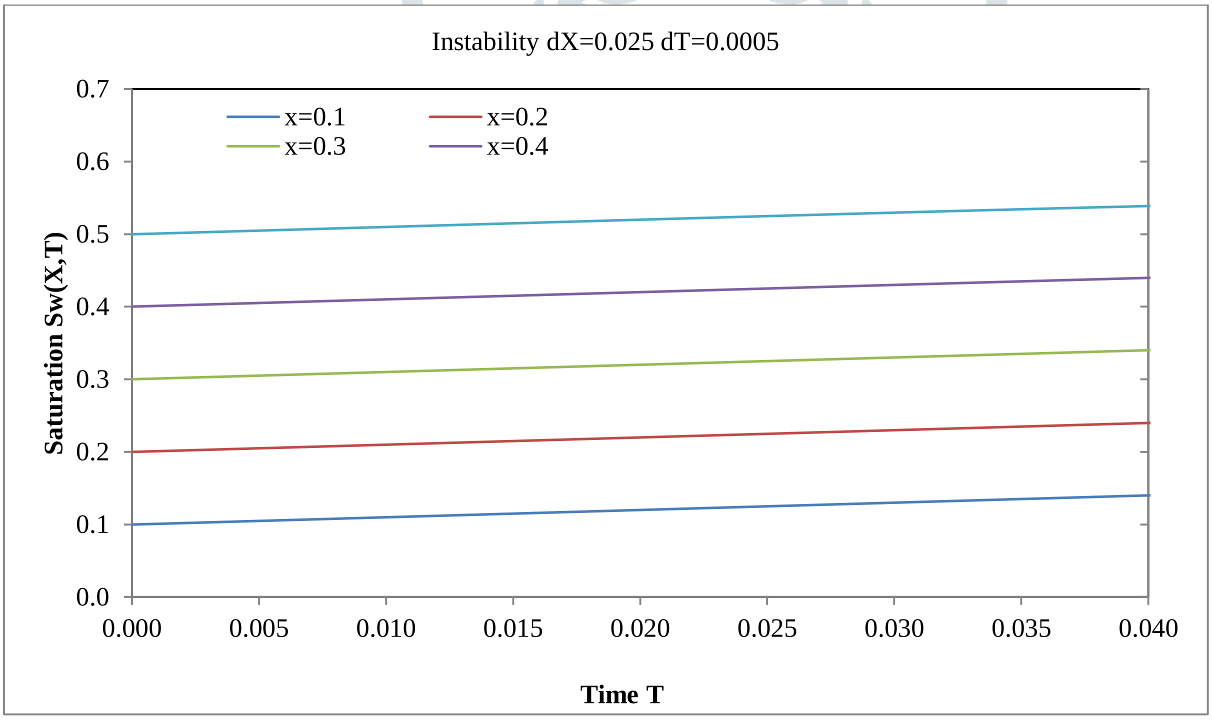
(1.21)

The numerical values obtained by using schemes given by the equations (1.21) to determine the saturation of injected water at different distance X (average length of schematic fingers) for given time T >0.

The tabular values are given for grid length dX=0.025, width dT =0.0005 and ratio r = 0.8
Table 1: The saturation $S_w(X,T)$ of water at different time $T > 0$ for the distances X varying from 0.1 to 0.5

By using scilab coding the above tabular values and the following figure 5 (i) and (ii) are obtained which represent the saturation of injected water Sw (X, T) vs. X and Sw (X, T) vs. T.





Conclusion:

The tabular values and graphical presentation obtained by using scilab coding for the finite difference scheme of the equation (1.17) by Crank-Nicolson method shows that the saturation of injected water is linearly increasing as distance X , also, it is linearly increasing for increasing time T for different length of the fingers. From the above discussion we have concluded that saturation of injected water is linearly increasing in instability phenomenon.

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