PROJECTILE MOTION WITH QUADRATIC DRAG

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Abstract : The intention of this paper provides a detailed study of motion of projectiles in quadratic drag along with the trajectory and range in a linear medium and also with path of projectile and its characteristics

***IndexTerms* - Motion, fundamental principles,path of projectile,characteristics, horizontal and vertical quadratic drag**

I. INTRODUCTION

The projectile motion is a form of motion experienced by an object or particle(projectile) that is thrown near the earth's surface and moves along a curved path under the action of gravity only. An object moving through fluid is influenced by resistant force(Drag) that acts oppositely to the relative motion of objects. Drag (sometimes called air resistance, a type of friction, or fluid resistance, another type of friction or fluid friction) is a force acting opposite to the relative motion of any object moving with respect to a surrounding fluid.Depending upon the characteristics of the flow--represented by a dimensionless quantity referred to as the Reynolds number--two different types of drag models are used: a linear drag and a quadratic drag model.

3.1 EQUATION OF MOTION

Consider a motion of a particle projected into the air in any direction and with any velocity.Such particle is called a Projectile.The two forces that act on the projectile are its weight and the resistance of air.For simplicity,we suppose the motion to take place within such a moderate distance from the surface of the earth that we can neglect the variations in the acceleration due to gravity.This means that g may be considered to be constant in magnitude throughout the motion of the projectile.Secondly,We shall neglect the resistance of the air and consider the motion to take in vacuum.

The angle of projection is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.The *velocity of projection* is the velocity with which the particle is projected.

The *trajectory* is the path which the particle describes.The *range* on a plane through the point of projection is the distance between the point of projection and the point where the trajectory meets the plane.The *time of flight* is the interval of the time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

3.2 TWO FUNDAMENTAL PRINCIPLES

We consider the horizontal and vertical components of the motion separately.The only force acting on the projectile is gravity and this acts vertically downwards.Hence by the Physical Independence of forces,it has no effect on the horizontal motion of the particle.So the horizontal velocity remains constant throughout the motion,as there is no force to cause any acceleration in that direction.On the other hand,the weight of the particle acting vertically downwards,will have its full effect on the vertical motion of the particle.The weight mg acting vertically downwards on a particle of mass m will produce an acceleration g vertically downwards .Hence the vertical component of the velocity will be subject to a retardation g.These two main principles will help us to study the projectile motion.

3.3 PATH OF A PROJECTILE

Let a particle be projected from O,with a velocity u at an angle α to the horizon.Take O as the origin,the horizontal and the upward vertical through O as of x and y respectively.The initial velocity u can be split into two components,which are $u\cos\alpha$ in the horizontal direction and $u\sin\alpha$ in the vertical direction.The horizontal component $u\cos\alpha$ is constant throughout the motion as there is no horizontal acceleration.The vertical component $u\sin\alpha$ is subject to an acceleration g downwards.

Let $P(x, y)$ be the position of the particle at time t secs,after projection.Then

x =horizontal distance described in t secs = $(u\cos\alpha)t$ ....(1)

y =vertical distance described in t secs = $(u\sin\alpha)t - \frac{1}{2}gt^2$ .....(2)

Eq.1and Eq.2 can be taken as the parametric equations of the trajectory.The equation to the path is got by eliminating t between them.

From Eq.1, $t = \frac{x}{u\cos\alpha}$ and putting this in Eq.2,we get

$$y = u\sin\alpha.\frac{x}{u\cos\alpha} - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

(ie) $$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \text{...........(3)}$$

Multiplying Eq.3 by $2u^2\cos^2\alpha$,

$$2u^2\cos^2\alpha.y = 2u^2\cos^2\alpha.x\frac{\sin\alpha}{\cos\alpha} - gx^2$$

(ie) $$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x = -\frac{2u^2\cos^2\alpha}{g}y$$

Or $$\left(x - \frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} - \frac{2u^2\cos^2\alpha}{g}y$$

$$= -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$$

Transfer the origin to the point

$$\left(\frac{u^2\sin\alpha\cos\alpha}{g},\frac{u^2\sin^2\alpha}{2g}\right)$$

The above equation then becomes

$$X^2=-\frac{2u^2\cos^2\alpha}{g}.Y..................(4)$$

(4) is clearly the equation to a parabola of latus rectum $\frac{2u^2\cos^2\alpha}{g}$, whose axis is vertical and downwards and whose vertex is the point $\left(\frac{u^2\sin\alpha\cos\alpha}{g},\frac{u^2\sin^2\alpha}{2g}\right)$

The latus rectum of the above parabola is

$$\frac{2u^2\cos^2\alpha}{g}=\frac{2}{g}.(u\cos\alpha)^2$$

$$=\frac{2}{g}(square of the horizontal velocity)$$

So the latus rectum (ie.the size of the parabola) is independent of the initial vertical velocity and depends only on the horizontal velocity.

3.4 CHARACTERISTICS OF THE MOTION OF A PROJECTILE

Let a particle be projected from O with velocity u at an angle α to the horizontal OX.Let A be the highest point of the path and C the point where it again meets the horizontal plane through O.Using the two fundamental principles,we can derive the following results relating to the motion of a projectile.

3.4.1 GREATEST HEIGHT ATTAINED BY A PROJECTILE

At A,the highest point,the particle will be moving only horizontally,having lost all its vertical velocity .Let $AB=h$ =the greatest height reached.Considering vertical motion separately,initial upward vertical velocity $=u\sin\alpha$ and the acceleration in this direction is $-g$ .The final vertical velocity at A is $=0$

Hence, $0=(u\sin\alpha)^2-2g.h$

(ie) $h=\frac{u^2\sin^2\alpha}{2g}$

(ie) the vertex of the parabola is the highest point of the path.

3.4.2 TIME TAKEN TO REACH THE GREATEST HEIGHT

Let T be the time from O to A.Then,in time T,the initial vertical velocity $u\sin\alpha$ is reduced to zero ,acted on by an acceleration $-g$. Hence $0 = u\sin\alpha - gT$

$$T = \frac{u\sin\alpha}{g}$$

3.4.3 THE RANGE ON THE HORIZONTAL PLANE THROUGH THE POINT OF PROJECTION

The time of flight is $t = \frac{2u\sin\alpha}{g}$ .During this time,the horizontal velocity remains constant and is equal to $u\cos\alpha$

Hence OC=horizontal distance described in time *t.*

$$= u\cos\alpha . t = u\cos\alpha . \frac{2u\sin\alpha}{g} = \frac{2u^2\sin\alpha\cos\alpha}{g}$$

Hence, the horizontal range R is,

$$= \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

The horizontal range can also be found by equation of the path,

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \,..........(1)$$

The equation to the x-axis is $y = 0$

Putting *y=0* in (1),we have,

$$x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} = 0$$

(ie) $x = 0$ or $x = \frac{2u^2\cos^2\alpha\tan\alpha}{g} = \frac{2u^2\sin\alpha\cos\alpha}{g}$

$x = 0$ corresponds to the point of projection and so the other value $\frac{2u^2\sin\alpha\cos\alpha}{g}$ gives horizontal range

3.5 Trajectory And Range In A Linear Medium

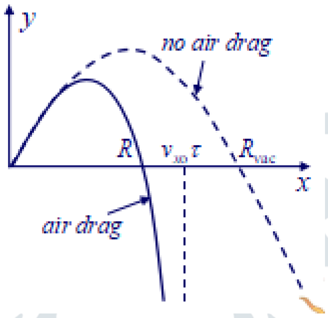
Now we can have the solutions for the motion in both x and y, in particular

$$x(t) = v_{x0}\tau(1-e^{-t/\tau})$$

$$y(t) = (v_{y0} - v_{ter})\tau(1-e^{-t/\tau})v_{ter}t$$

where the sign of v_{ter} is reversed so that positive y is vertically upward. We'd like to plot the trajectory, and this is easier if we eliminate t in the above equations to give one equation of y as a function of x. This gives

$$y = \left[\left(\frac{v_{y0}+v_{ter}}{v_{x0}}\right)\right]x + v_{ter}\tau\ln\left[\frac{1-x}{v_{x0}\tau}\right]$$



FIGURE

3.6 QUADRATIC DRAG

The drag equation with a constant drag coefficient gives the force experienced by an object moving through a fluid at relatively large velocity(I.e. high Reynolds number,Re>~1000).This is also called **quadratic drag.**

3.6.1 HORIZONTAL AND VERTICAL MOTION WITH QUADRATIC DRAG:

The equation of motion for a projectile depending on quadratic drag,

$$mr'' = mg - cv^2\hat{v}$$

$$= mg - cv\,v \qquad (1)$$

resolves into its horizontal and vertical components(with *y* measured vertically upward)to give

$$m\hat{v}_x = -c\sqrt{\left(v_x{}^2 + v_y{}^2 v_x\right)}$$

$$m\hat{v}_y = -mg - c\sqrt{\left(v_x{}^2 + v_y{}^2 v_y\right)} \qquad (2)$$

These are two differential equations for the two unknown functions $v_x(t)$ and $v_y(t)$,but each equation involves both v_x and v_y . In particular, neither equation is the same as for an object that changes only in the *x* direction or only in the y direction. This means that we cannot solve these two equation by simply combining together our two separate solutions for horizontal and vertical motion. Worse still, it turns out that the two Eq.2 cannot be solved analytically at all. The only way to solve them is numerically, which we can only do for some numerical initial conditions (that is, specified values of the initial position and velocity). This means that we can't find the general solution; all we can do numerically is to find the particular solution corresponding to any chosen initial conditions.

This examples illustrates some of the general features of projectile motion with a quadratic drag force.Although we cannot solve analytically the equations of motion Eq.2 for this problem,we can use the equations to prove various general properties of the trajectory.For example,we noticed that the baseball reached a lower maximum height,and did so sooner,than it would have in a vacuum.It is easy to prove that this will always be the case:As long as the projectile is moving in upward direction $(v_y > 0)$,,the force of air resistance has a downward y component.Thus the downward acceleration is greater than *g*(its value in vacuum).Therefore a graph of v_y against *t* slopes down from v_{y0} more quickly than it would in vacuum.This proves that v_y reaches zero sooner than it would in vacuum,and that the ball travels less distance (in the direction)before reaching the high point.That is,the ball's high point occurs sooner and is lower than it would be in a vacuum.

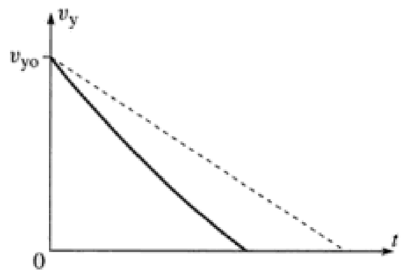


FIGURE:*Graph of v_y against t for a projectile that is thrown upward $(v_{y0} > 0)$ and is subject to a quadratic resistance(solid curve).The dashed line(slope =g) is the corresponding graph when there is no air resistance.The projectile moves upward until it reaches its maximum height when $v_y = 0$.During this time,the drag force is downward and the downward acceleration is always greater than g.Therefore,the curve slopes more steeply than the dashed line,and the projectile reaches its high point sooner than it would in a vacuum.Since the area under the curve is less than that under the dashed line,the projectile's maximum height is less than it would be in a vacuum.*

First,it is easy to convince that once the ball starts moving downward,it continues to accelerate downward,with v_y approaching $-v_{ter}$ as t→∞.At the same time v_x continues to decrease and approaches zero.Thus the square root in both of the Eq.2 approaches v_{ter}.In particular,when *t* is large,the equation of v_x can be approximated by

$$\hat{v}_x \approx \left(\frac{-cv_{ter}}{m}\right)v_x = -kv_x(say)$$

The solution of this equation is,of course,an exponential function, $v_x = Ae^{-kt}$,and we see that v_x approaches zero very rapidly (exponentially)as t→∞.This guarantees that *x*,which is the integral of v_x ,

$$x(t) = \int_0^t v_x(t')dt'$$

approaches a finite limit as t→∞ and the trajectory has a finite vertical asymptote as claimed.

II. Acknowledgment

The discussion of a projectile motion under air resistance in quadratic drag. Along with path of projectile motion and its characteristics,this case study will be interesting for students of undergraduate physics and applied mathematics courses.

References

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