ANALYTICAL SOLUTION OF BLACK-SCHOLES EUROPEAN OPTION PRICING PAYING CONTINUOUS DIVIDENDS THROUGH ADOMAIN DECOMPOSITION METHOD

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Abstract:

In this paper, the Adomian decomposition method has been applied to obtain the analytical solution of Black-Scholes partial differential equation for European options over an asset that pays continuous dividends. The Analytical solution is obtained as a convergent power series in which each term is calculated easily. The solution is obtained without any discretization and hence the computation is reduced to a greater extent.

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I. Introduction:

An option is a financial derivative whose value depends on the price of another asset. Thus, options are financial contracts that provide an option to buy or sell a certain underlying asset at a specific price and at a certain fixed future date. Options are divided into two categories based on their purchasing and selling. The right to buy an option is called a call option and the right to sell it is known as put option. The wide use of options has gained a lot of attention in financial markets both practically and theoretically. Options are used to control or account the risk caused by the movement in stock prices by creating portfolios and by hedging assets. Fischer Black and Myron Scholes in 1973 [1] derived a formula for pricing both European and American options. The Black-Scholes formula is a second order linear partial differential equation. This formula is popularly known as Black-Scholes model and it proved to be an effective model for pricing different kinds of options. The Black-Scholes option pricing model is based on certain assumptions and one of the key assumptions is that the underlying asset does not pay any dividends during the life time of the option. Merton [2] extended the Black-Scholes option pricing model to underlying assets that pay a continuous dividend yield during the life time of the option and derived the modified Black-Scholes equation and the modified Black-Scholes formulae for both European call and put options. The Black-Scholes equation for European options paying continuous dividends is given by the equation;

$$\frac{\partial f}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}+(r-D)S\frac{\partial f}{\partial S}-rf=0 \qquad (1.1)$$

Where $f(S,t)$ is the option price or option premium at asset price S and at time t, $S(t)$ is the asset price at time t, σ represents the volatility, r is the risk free interest rate and D is the dividend yield.
The pay off functions for European call and put options are given by;

$$f_c(S,t)=\max(S-K,0) \text{ and } f_p(S,t)=\max(K-S,0) \qquad (1.2)$$

Where K denotes the strike price.
During the last few decades, many researchers have used many methods to obtain the various possible solutions of Black-Scholes model. The important methods include finite difference method [3,4], homotopy

perturbation method [5,6,7], variational iteration method [8], homotopy analysis method [9], differential transform method [10], Adomian decomposition method [11].

In this article we will present the Adomian decomposition method and apply it to Black Scholes model for European option pricing paying continuous dividends. The Adomian decomposition technique is used to obtain the analytical solution of linear and non linear differential equations. In this method the unknown function is decomposed in an infinite series $\sum_{n=0}^{\infty} u_n$ and the non linear term is also decomposed in another series $\sum_{n=0}^{\infty} A_n$ where $A_n's$ are Adomian polynomials [12]. This method was developed by George Adomian in 1980's to obtain different solutions of linear and non linear differential equations characterizing stochastic systems [13].

The rest part of the paper is outlined as: in Section II, we have given the analysis of Adomian decomposition method. The description of Black-Scholes model and European options are given in section III. In Section IV, the solution of Black-Scholes equation paying continuous dividends is carried out through Adomian decomposition method. Finally, the conclusion is given in Section V.

II. Analysis of Adomian decomposition method:

Consider the following non linear differential equation;

$$L_\tau u(x,\tau) + Ru(x,\tau) + Nu(x,\tau) = g(x,\tau) \qquad (2.1)$$

Where $L_\tau = \frac{\partial}{\partial \tau}$, R is the linear remainder operator, N represents a non linear operator and g is a non homogenous term independent of u.

Solving equation (2.1) for $L_\tau u(x,\tau)$ we have

$$L_\tau u(x,\tau) = g(x,\tau) - Ru(x,\tau) - Nu(x,\tau) \qquad (2.2)$$

Applying ${L_\tau}^{-1}$ on both sides of equation (2.2), as L_τ is invertible, we get

$${L_\tau}^{-1} L_\tau u(x,\tau) = {L_\tau}^{-1} g(x,\tau) - {L_\tau}^{-1} Ru(x,\tau) - {L_\tau}^{-1} Nu(x,\tau) \qquad (2.3)$$

We get,

$$u(x,\tau) = C + {L_\tau}^{-1} g(x,\tau) - {L_\tau}^{-1} Ru(x,\tau) - {L_\tau}^{-1} Nu(x,\tau) \qquad (2.4)$$

Where C is the constant of integration satisfying $L_\tau C = 0$

The ADM presumes a decomposition solution in an infinite series form $u(x,\tau)$ given as

$$u(x,\tau) = \sum_{i=0}^{\infty} u_i(x,\tau) \qquad (2.5)$$

The decomposition of non linear term is given as

$$Nu(x,\tau) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots u_n) \qquad (2.6)$$

Where $A_n's$ are Adomian polynomials.

Substituting (2.5) and (2.6) in (2.4)

$$\sum_{i=0}^{\infty} u_i(x,\tau) = C + {L_\tau}^{-1} g(x,\tau) - {L_\tau}^{-1} R \sum_{i=0}^{\infty} u_i(x,\tau) - {L_\tau}^{-1} \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots u_n) \qquad (2.7)$$

The solution through this method is obtained as

$$u_0(\tau) = C + {L_\tau}^{-1} g(x,\tau) \qquad (2.8)$$

$$u_{n+1}(\tau) = -{L_\tau}^{-1} Ru_i(x,\tau) - {L_\tau}^{-1} A_n(u_0, u_1, \dots u_n) \qquad (2.9)$$

The approximate solution of (2.1), using (2.9) is given as;

$$u(x,\tau) = \sum_{i=0}^{\infty} u_i(x,\tau) \qquad (2.10)$$

A large number of linear and non linear equations have been solved through ADM [14]. This method requires less computation as compared to other methods [15]. This method obtains a series solution with easily computed terms. In general the decomposition of the series solution converges very quickly through this method. Hence only a few terms are required for approximation [16,17].

III. Black-Scholes model and European options:

The Black-Scholes model for European options paying continuous dividends is given by [2];

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}+(r-D)S\frac{\partial f}{\partial S}-rf=0 \qquad (3.1)$$

Where $f(S,t)$ is the option price or option premium at asset price S and at time t, $S(t)$ is the asset price at time t, , σ represents the volatility, r is the risk free interest rate and D is the dividend yield.

The final condition for call options is

$$f(S,t)=\max(S-K,0) \qquad (3.2)$$

And the final condition for put options is

$$V(S,t)=\max(K-S,0) \qquad (3.3)$$

Where K denotes the strike price.

By changing certain variables, the Black-Scholes equation gets transformed into a standard boundary value problem for the heat equation.

Put, $S=e^x,\ t=T-\frac{2\tau}{\sigma^2}, f(S,t)=v(x,\tau)=v(\log S,\frac{\sigma^2}{2}(T-t))$

The partial derivatives of f with respect to S and t expressed in terms of partial derivatives of v in terms of x and τ are:

$$\frac{\partial f}{\partial t}=-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}$$
$$\frac{\partial f}{\partial S}=\frac{1}{S}\frac{\partial v}{\partial x}$$
$$\frac{\partial^2 f}{\partial S^2}=-\frac{1}{S^2}\frac{\partial v}{\partial x}+\frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}$$

Putting above three expressions in equation (3.1) we get

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k_2-1)\frac{\partial v}{\partial x}-k_1v \qquad (3.4)$$

where, $k_1=\frac{2r}{\sigma^2}$ $and\ k_2=\frac{2(r-D)}{\sigma^2}$

The final condition becomes

$$v(x,0)=\begin{cases}\max(e^x-1,0), & for\ call\ options\\ \max(1-e^x,0), & for\ put\ options\end{cases} \qquad (3.5)$$

IV. Solution of Black-Scholes equation through ADM:

Consider equation (3.4)

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k_2-1)\frac{\partial v}{\partial x}-k_1v \qquad (4.1)$$

Subject to $v(x,0)=\max(e^x-1,0)$ (4.2)

To solve equation (4.1) and (4.2) through Adomian decomposition method we have;

$$v(x,\tau)=\max(e^x-1,0)-\int_0^\tau -\frac{\partial^2 v}{\partial x^2}-(k_2-1)\frac{\partial v}{\partial x}+k_1v\,d\tau \qquad (4.3)$$

The solution by Adomian decomposition method is given in the following way;

$$v_0=\max(e^x-1,0) \qquad (4.4)$$
$$v_{n+1}=-\int_0^\tau A_n(v_0,v_1,v_2,\dots v_n)d\tau \qquad n=0,1,2,3,\dots \qquad (4.5)$$

To find $A_n's$ let $v_\lambda=\sum_{n=0}^{\infty}\lambda^n v_n$ and computing

$$N(v)=-\frac{\partial^2 v}{\partial x^2}-(k_2-1)\frac{\partial v}{\partial x}+k_1v \qquad (4.6)$$

$$\begin{cases}A_0(v_0)=-\frac{\partial^2 v_0}{\partial x^2}-(k_2-1)\frac{\partial v_0}{\partial x}+k_1v_0\\ A_1(v_0,v_1)=-\frac{\partial^2 v_1}{\partial x^2}-(k_2-1)\frac{\partial v_1}{\partial x}+k_1v_1\\ \quad\vdots \qquad\qquad\qquad\qquad \vdots\\ A_n(v_0,v_1,\dots v_n)=-\frac{\partial^2 v_n}{\partial x^2}-(k_2-1)\frac{\partial v_n}{\partial x}+k_1v_n\end{cases} \qquad (4.7)$$

In this way

$$v_0(x,\tau)=\max(e^x-1,0) \qquad (4.8)$$
$$v_1(x,\tau)=-\int_0^\tau A_0(v_0)d\tau \qquad (4.9)$$
$$\Rightarrow v_1(x,\tau)=-A_0(v_0)\tau$$

$$\Rightarrow v_1(x,\tau) = [\max(e^x,0) + (k_2-1)\max(e^x,0) - k_1\max(e^x-1,0)]\tau$$

$$\Rightarrow v_1(x,\tau) = k_2\tau\max(e^x,0) - k_1\tau\max(e^x-1,0) \quad (4.10)$$

$$\frac{\partial}{\partial x}v_1(x,\tau) = (k_2\tau - k_1\tau)\max(e^x,0) \quad (4.11)$$

$$v_2(x,\tau) = -\int_0^\tau A_1(v_0,v_1)d\tau \quad (4.12)$$

$$\Rightarrow v_2(x,\tau) = -\int_0^\tau (-\frac{\partial^2 v_1}{\partial x^2} - (k_2-1)\frac{\partial v_1}{\partial x} + k_1 v_1)d\tau$$

$$\Rightarrow v_2(x,\tau) = \int_0^\tau \left(\frac{\partial^2 v_1}{\partial x^2} + (k_2-1)\frac{\partial v_1}{\partial x} - k_1 v_1\right)d\tau$$

$$\Rightarrow v_2(x,\tau) = \left[{k_2}^2\max(e^x,0) - 2k_1k_2\max(e^x,0) + {k_1}^2\max(e^x-1,0)\right]\int_0^\tau \tau d\tau$$

$$\Rightarrow v_2(x,\tau) = \frac{k_2^2\tau^2}{2}\max(e^x,0) - 2k_1k_2\frac{\tau^2}{2}\max(e^x,0) + \frac{k_1^2\tau^2}{2}\max(e^x-1,0) \quad (4.13)$$

The derivative of $v_2(x,\tau)$ is given by

$$\frac{\partial}{\partial x}v_2(x,\tau) = \frac{k_2^2\tau^2}{2}\max(e^x,0) - k_1k_2\tau^2\max(e^x,0) + \frac{k_1^2\tau^2}{2}\max(e^x,0) \quad (4.14)$$

$$\frac{\partial}{\partial x}v_2(x,\tau) = \frac{(k_2\tau - k_1\tau)^2}{2}\max(e^x,0) \quad (4.15)$$

$$v_3(x,\tau) = -\int_0^\tau A_2(v_0,v_1,v_2)d\tau \quad (4.16)$$

$$\Rightarrow v_3(x,\tau) = -\int_0^\tau \left(-\frac{\partial^2 v_2}{\partial x^2} - (k_2-1)\frac{\partial v_2}{\partial x} + k_1 v_2\right)d\tau$$

$$\Rightarrow v_3(x,\tau) = \int_0^\tau \left(\frac{\partial^2 v_2}{\partial x^2} + (k_2-1)\frac{\partial v_2}{\partial x} - k_1 v_2\right)d\tau$$

$$\Rightarrow v_3(x,\tau) = \left[\left(\frac{{k_2}^3}{2} + \frac{3{k_1}^2k_2}{2} - \frac{3{k_2}^2k_1}{2}\right)\max(e^x,0) - \frac{{k_1}^3}{2}\max(e^x-1,0)\right]\int_0^\tau \tau^2 d\tau$$

$$\Rightarrow v_3(x,\tau) = \left[\left(\frac{{k_2}^3}{6} + \frac{3{k_1}^2k_2}{6} - \frac{3{k_2}^2k_1}{6}\right)\tau^3\max(e^x,0) - \frac{{k_1}^3}{6}\tau^3\max(e^x-1,0)\right]$$

The derivative of $v_3(x,\tau)$ is given by

$$\frac{\partial}{\partial x}v_3(x,\tau) = \left[\left(\frac{{k_2}^3}{6} + \frac{3{k_1}^2k_2}{6} - \frac{3{k_2}^2k_1}{6} - \frac{{k_1}^3}{6}\right)\tau^3\max(e^x,0)\right]$$

$$\Rightarrow \frac{\partial}{\partial x}v_3(x,\tau) = \frac{(k_2\tau - k_1\tau)^3}{6}\max(e^x,0) \quad (4.17)$$

Working in this way we get,

$$\frac{\partial}{\partial x}v_n(x,\tau) = \frac{(k_2\tau - k_1\tau)^n}{n!}\max(e^x,0) \quad (4.18)$$

Hence the solution of the problem is given by

$$v(x,\tau) = \max(e^x-1,0) - \int_0^\tau -\frac{\partial^2 v}{\partial x^2} - (k_2-1)\frac{\partial v}{\partial x} + k_1 v\, d\tau \quad (4.19)$$

$$\Rightarrow v(x,\tau) = \max(e^x-1,0) + \sum_{n=1}^{\infty} v_n(x,\tau) \quad (4.20)$$

When differentiating equation (4.20) partially with respect to x, we have

$$\frac{\partial}{\partial x}v(x,\tau) = \max(e^x,0) + \sum_{n=1}^{\infty}\frac{\partial}{\partial x}v_n(x,\tau) \quad (4.21)$$

$$\frac{\partial}{\partial x}v(x,\tau) = \max(e^x,0) + \sum_{n=1}^{\infty}\frac{(k_2\tau-k_1\tau)^n}{n!}\max(e^x,0) \quad (4.22)$$

On integrating equation (4.22) with respect to x we have,

$$v(x,\tau) = \max(e^x,0) + \sum_{n=1}^{\infty}\frac{(k_2\tau-k_1\tau)^n}{n!}\max(e^x,0) \quad (4.23)$$

Hence, equation (4.23) is the exact solution of equation(4.1).

V. Conclusion

The Black-Scholes model is one of the important and useful models for the pricing of options in financial markets. In this article we have given a brief discussion of Adomian decomposition method and applied this method to obtain the exact solution of Black-Scholes equation for European options paying continuous dividends. The results obtained here prove the efficiency and efficacy of the proposed method. Therefore this method can be used and applied successfully to other linear and non linear differential equations arising in financial mathematics. The same algorithm can be applied to put options also.

VI. References

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