FUZZY IDEALS AND WEAK IDEALS IN BE – ALGEBRA

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ABSTRACT In this paper, we use the notion of fuzzy point to study some basic algebraic structures such as BE – algebras and ideals. Then we clarify the links between the fuzzy point approach and the classical fuzzy approach.

KEY WORDS Fuzzy algebra, Sub algebra, BE algebra, Weak ideal, Commutative ideal.

INTRODUCTION

In [4,7] some transfer theorems for fuzzy groups and fuzzy semigroups were established. In this paper, we apply those results to BE – algebras. The concept of fuzzy sets was introduced by Zadeh [9]. This concept has been applied to BE – algebras by Xi [8]. In this paper, given a BE – algebra (X,∗ ,1) and a fuzzy subset A on X, we construct the set $(\tilde{X}, *)$ of all fuzzy points on X and the subset $\tilde{A}$ of $\tilde{X}$. Then we establish and similarities between some properties of A and $\tilde{A}$.

PRELIMINARIES

An algebra (X,∗ ,1) of type (2,0) is called a BE- algebra if for all x, y and z in X

1. $x * x = 1$
2. $x * 1 = 1$
3. $1 * x = x$
4. $x * (y * z) = y * (x * z)$

In X, a binary relation ‘’ $\leq$ ” is defined by x ≤ y if and only if $x * y = 1$.

The following properties also hold in BE algebra.

1. $x * y = 1$ and $y * x = 1$ imply x = y
2. $x * y = 1$ and $y * z = 1$ imply $x * z = 1$
3. $x * y = 1$ implies $(x * z) * (y * z) = 1$ and $(z * y) * (z * x) = 1$
4. $(x * y) * z = (x * z) * y$
5. $(x * y) * x = 1$
6. $x * \big(x * (x * y)\big) = x * y$
7. $(x * y) * z = 1$ implies $(x * z) * y = 1$
8. $[(x * y) * (y * z)] * (x * y) = 1$
9. $\big[\big((x * z) * z\big) * (y * z)\big] * [(x * y) * z] = 1$
10. $(x * z) * \big(x * (x * z)\big) = (x * z) * z$
11. $\Big[x * \big(y * (y * x)\big) * \Big(y * \Big(x * \big(y * (y * x)\big)\Big)\Big)\Big] * (x * y) = 1$

12. $((x*y)*(x*z))*(z*y)=1$
13. $(x*(x*y))*y=1$

ALGEBRAIC STRUCTURE OF THE SET OF FUZZY POINTS IN BE ALGEBRAS

Let (X,$*$, 1) be a BE algebra. A fuzzy set A in X is a map A: X → [0,1]. If ξ is the family of all fuzzy sets in X, $x_\lambda \in \xi$ is a fuzzy point if and only if $x_\lambda(y) = \lambda$ when x = y; and $x_\lambda(y) = 1$ when $x \neq y$. We denote by $\tilde{X} = \{x_\lambda | x \in X, \lambda \in (0,1]\}$ the set of all fuzzy points on X and define a binary operation on $\tilde{X}$ as follows: $x_\lambda * y_\mu = (x*y)_{\min(\lambda,\mu)}$.

It is easy to verify that $(\tilde{X},*)$ satisfies the following conditions: for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

BE (1') $(x_\lambda * x_\lambda) = 1_\lambda$

BE (2′) $x_\lambda * 1_\mu = 1_{\min\{\lambda,\mu\}}$

BE (3′) $1_\lambda * x_\mu = x_{\min\{\lambda,\mu\}}$

BE (4′) $x_\lambda * (y_\mu * z_\alpha) = y_\mu * (x_\lambda * z_\alpha)$

We can also establish the following conditions for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

(1′) $x_\lambda * y_\mu = 1_{\min\{\lambda,\mu\}}$ and $y_\mu * z_\alpha = 1_{\min\{\mu,\alpha\}}$ imply $x_\lambda * z_\alpha = 1_{\min\{\lambda,\alpha\}}$

(2′) $x_\lambda * y_\mu = 1_{\min\{\lambda,\mu\}}$ implies $(x_\lambda * z_\alpha) * (y_\mu * z_\alpha) = 1_{\min\{\lambda,\mu,\alpha\}}$

and $(z_\alpha * y_\mu) * (z_\alpha * x_\lambda) = 1_{\min\{\lambda,\mu,\alpha\}}$

(3′) $(x_\lambda * y_\mu) * z_\alpha = (x_\lambda * z_\alpha) * y_\mu$

(4′) $(x_\lambda * y_\mu) * x_\lambda = 1_{\min\{\lambda,\mu\}}$

(5′) $x_\lambda * (x_\lambda * (x_\lambda * y_\mu)) = (x_\lambda * y_\mu)$

(6′) $(x_\lambda * y_\mu) * z_\alpha = 1_{\min\{\lambda,\mu,\alpha\}}$ implies $(x_\lambda * z_\alpha) * y_\mu = 1_{\min\{\lambda,\mu,\alpha\}}$

(7′) $[(x_\lambda * y_\mu) * (y_\mu * z_\alpha)] * (x_\lambda * y_\mu) = 1_{\min\{\lambda,\mu,\alpha\}}$

(8′) $[((x_\lambda * z_\alpha) * z_\alpha) * (y_\mu * z_\alpha)] * [(x_\lambda * y_\mu) * z_\alpha] = 1_{\min\{\lambda,\mu,\alpha\}}$

(9′) $(x_\lambda * z_\alpha) * (x_\lambda * (x_\lambda * z_\alpha)) = (x_\lambda * z_\alpha) * z_\alpha$

(10′) $\{[x_\lambda * (y_\mu * (y_\mu * x_\lambda))] * [y_\mu * (x_\lambda * (y_\mu * (y_\mu * x_\lambda)))]\} * x_\lambda * y_\mu = 1_{\min\{\lambda,\mu\}}$

(11′) $((x_\lambda * y_\mu) * (x_\lambda * z_\alpha)) * (z_\alpha * y_\mu) = 1_{\min\{\lambda,\mu,\alpha\}}$

(12′) $(x_\lambda * (x_\lambda * y_\mu)) * y_\mu = 1_{\min\{\lambda,\mu\}}$

We also recall that: if A is a fuzzy subset of a BE – algebra X, then we have the following:

$\tilde{A} = \{x_\lambda \in \tilde{X}\ |A(x) \geq \lambda, \lambda \in (0,1]\}$, and for any, $\lambda \in (0,1]$ $\tilde{X}_\lambda = \{x_\lambda | x \in X\}$, and $\tilde{A}_\lambda = \{x_\lambda \in \tilde{X}\ |A(x) \geq \lambda\}$. Hence $\tilde{X}_\lambda \subseteq \tilde{X},\ \tilde{A} \subseteq \tilde{X},\ \tilde{A}_\lambda \subseteq \tilde{A},\ \tilde{A}_\lambda \subseteq \tilde{X}_\lambda$.

1.WEAK IDEAL

Definition 1.1

A non-empty subset I of BE – algebra X is called an ideal if it satisfies

1. $1 \in I$
2. $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 1.2

A fuzzy subset A of a BE – algebra X is a fuzzy subalgebra if and only if for any $x, y \in X$,

$A(x * y) \geq \min(A(x), A(y))$.

Definition 1.3

$\tilde{A}$ is a subalgebra of $\tilde{X}$ if and only if for any $x_\lambda, y_\mu \in \tilde{A}$ we have $x_\lambda, y_\mu \in \tilde{A}$.

Theorem 1.1

Let A be a fuzzy subset of a BE – algebra X. Then the following conditions are equivalent:

1) A is a fuzzy subalgebra of X.
2) For any $\lambda \in (0,1]$, $\tilde{A}_\lambda$ is a subalgebra of $\tilde{X}$.
3) For any $t \in (0,1]$, the t-level subset $A^t = \{x \in X | A(x) \geq t\}$ is a subalgebra of X when $A^t \neq \emptyset$.
4) $\tilde{A}$ is a subalgebra of $\tilde{X}$.

Proof:

1)⟹2) Let $x_\lambda, y_\mu \in \tilde{A}_\lambda$. Since A is a fuzzy subalgebra, $A(x * y) \geq \min(A(x), A(y)) \geq \lambda$, then $x_\lambda * y_\mu = (x * y)_\lambda \in \tilde{A}_\lambda$.

2)⟹3) Let x, y ∈ A^t. $\tilde{A}_t$ is a subalgebra, so we have $(x * y)_t = x_t * y_t \in \tilde{A}_t$.

Hence $x * y \in A^t$.

3)⟹4) Let $x_\lambda, y_\mu \in \tilde{A}$ and t = min (λ, μ). Then A(x)≥ $\lambda \geq t$ and A(y)≥ $\mu \geq t$, so x, y ∈ A^t. Since A^t is a subalgebra, $x * y \in A^t$ so that $x_\lambda * y_\mu = (x * y)_t \in \tilde{A}$.

4)⟹1) Let x, y ∈ X and t = min(A(x), A(y)). Then $x_t, y_t \in \tilde{A}$. Because $\tilde{A}$ is a subalgebra, so we have $(x * y)_t = x_t * y_t \in \tilde{A}$, hence $A(x * y) \geq t =$ min(A(x), A(y)).

Definition 1.4

A fuzzy subset A of a BE – algebra X is a fuzzy ideal if and only if :

a) for any $x \in X$, A (1) ≥ A(x)
b) for any x, y∈ X, $A(x) \geq \min(A(x * y), A(y))$.

Definition 1.5

$\tilde{A}$ is a weak ideal of $\tilde{X}$ if and only if

a) for any $\upsilon \in Im(A)$, $1_\upsilon \in \tilde{A}$,
b) for any $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$, $X_{\min(\lambda,\mu)} \in \tilde{A}$ holds.

Remark 1.1: Any weak ideal $\tilde{A}$ has the following property: $x_\lambda * y_\mu = 1_{\min\{\lambda,\mu\}}$ and $y_\mu \in \tilde{A}$ imply $x_{\min\{\lambda,\mu\}} \in \tilde{A}$. Clearly, let $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu = 1_{\min\{\lambda,\mu\}}$ and $y_\mu \in \tilde{A}$.

$y_\mu \in \tilde{A}$ implies that A(y)$\geq \mu$. Let A(y) = α. Using definition 1.5(a) we obtain $1_\alpha \in \tilde{A}$. So

A (1)$\geq \alpha$. But $\alpha = A(y) \geq \mu \geq \min(\lambda, \mu)$. So $1_{\min\{\lambda,\mu\}} \in \tilde{A}$. Using definition 1.5 (b), we obtain $x_{\min\{\lambda,\mu\}} \in \tilde{A}$.

Now we discuss the relation between subalgebra and weak ideal. First of all, let us establish the following:

Lemma 1.1: If $\check{A}$ is a subalgebra of $\tilde{X}$, then for any $\lambda \in Im(A)$ $1_\lambda \in \tilde{A}$.

Proof:

Let $\lambda \in Im(A)$ and take x in X such that A(x) = λ, then $x_\lambda \in \tilde{A}$. $\tilde{A}$ is a subalgebra and BE (1') imply $(x_\lambda * x_\lambda) = 1_\lambda \in \tilde{A}$.

Corollary: If A is a fuzzy subalgebra then for any x ∈ X, A(1) ≥ A(x).

Lemma 1.2: Let A be a fuzzy subalgebra of X and $\lambda, \mu \in (0,1]$ such that $\lambda \geq \mu$. Then

a) If $x_\lambda \in \tilde{A}$, then $x_\mu \in \tilde{A}$,
b) If $x_\lambda \in \tilde{A}$, then $1_\mu \in \tilde{A}$.

Proof:

a) $x_\lambda \in \tilde{A}$ implies $A(x) \geq \lambda$. Since $\lambda \geq \mu$ we obtain $A(x) \geq \mu$. So $x_\mu \in \tilde{A}$.
b) $x_\lambda \in \tilde{A}$ implies $A(x) \geq \lambda$. Since A is a fuzzy subalgebra, A(1) ≥ A(x)≥ $\lambda \geq \mu$ and $1_\mu \in \tilde{A}$.

Theorem 1.2: Any weak ideal $\tilde{A}$ is a subalgebra.

Proof:

Let $x_\lambda, y_\mu \in \tilde{A}$. $y_\mu \in \tilde{A}$ implies that A(y) ≥ μ. Let A(y) = α. Using definition 1.5 (a), we obtain $1_\alpha \in \tilde{A}$ such that A(1) ≥ α. But $\alpha = A(y) \geq \mu \geq \min(\lambda, \mu)$. So $1_{\min\{\lambda,\mu\}} \in \tilde{A}$. BE (4'), $(x_\lambda * y_\mu) * x_\lambda = 1_{\min\{\lambda,\mu\}}$. Using definition 1.5 (b), we obtain $x_\lambda * y_\mu \in \tilde{A}$.

Theorem 1.3: Suppose that $\tilde{A}$ is a subalgebra of $\tilde{X}$. Then the following conditions are equivalent:

1) A is a fuzzy ideal.
2) If $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ with y_μ and $z_\alpha \in \tilde{A}$, then $x_{\min\{\lambda,\mu,\alpha\}} \in \tilde{A}$
3) for any t ∈ (0,1], the t-level subset $A^t = \{x \in X | A(x) \geq t\}$ is an ideal when $A^t \neq \emptyset$.
4) If $x_\lambda, y_\mu \in \tilde{A}$ and $(z_\alpha * y_\mu) * x_\lambda = 1_{\min\{\lambda,\mu,\alpha\}}$, then $z_{\min\{\lambda,\mu,\alpha\}} \in \tilde{A}$.
5) for any x, y, z in X, the inequality $x * y \leq z$ implies A(x) ≥ $\min(A(y), A(z))$.
6) $\tilde{A}$ is a weak ideal.

Proof:

1) $\Longrightarrow$ 2) Let $z_\alpha, y_\mu \in \tilde{A}$ such that $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$. Since A is a fuzzy ideal, we have $A(x) \geq \min(A(x*y), A(y))$ and $A(x*y) \geq \min(A((x*y)*z), A(z))$. So A(x) ≥ $\min(\lambda, \mu, \alpha)$. Hence $x_{\min\{\lambda,\mu,\alpha\}} \in \tilde{A}$.

2) $\Longrightarrow$ 3) a) Let x ∈ A^t. Then A(x) ≥ t. Since A is a fuzzy subalgebra, A(1) ≥ A(x). So A(1) ≥ A(x)≥ t, 1 ∈ A^t.

b) Let x, y in X such that $x*y \in A^t$ and $y \in A^t$. $y \in A^t$ implies A(y) ≥ t. Since A is a fuzzy subalgebra, A(1) ≥ A(y). So A(1) ≥ A(y)≥ t, $1_t \in \tilde{A}$. $x*y \in A^t$ implies A($x*y$)≥ t. So $(x*y)_t \in \tilde{A}$. Since $1_t \in \tilde{A}$ and $(x_t * y_t) * 1_t$ = $(x*y)_t \in \tilde{A}$, using the hypothesis, we obtain $x_t \in \tilde{A}$, so x ∈ A^t.

3) $\Longrightarrow$ 4) If $x_\lambda, y_\mu \in \tilde{A}$ with $(z_\alpha * y_\mu) * x_\lambda = 1_{\min\{\lambda,\mu,\alpha\}}$, we have (z∗ $y) * x = 1$. Let t = $\min(\lambda, \mu, \alpha)$, since A^t is an ideal, 1∈ A^t and because x, y ∈ A^t we obtain z∈ A^t. So z_t= $z_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

4) $\Longrightarrow$ 5) Let x, y, z in X such that x ∗ $y \leq z$ and $\mu = A(y), \alpha = A(z)$. Since x ∗ $y \leq z$, we have $\left(x_{\min(\mu,\alpha)*} y_\mu\right) * z_\alpha = 1_{\min(\mu,\alpha)}$. Using the hypothesis, we obtain $x_{\min(\mu,\alpha)} \in \tilde{A}$. So A(x) ≥ $\min(\mu, \alpha) = \min(A(y), A(z))$.

5) $\Longrightarrow$ 6) a) By lemma 1.1, it is clear that for any $v \in Im(A)$, $1_v \in \tilde{A}$.

b) for $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$. We have $A(x*y) \geq \min(\lambda, \mu)$ and A(y) ≥ μ. Since x∗ (x ∗ $y) \leq y$, it follows from the hypothesis that $A(x) \geq \min(A(x*y), A(y)) \geq \min(\lambda, \mu)$, so that $x_{\min(\mu,\alpha)} \in \tilde{A}$.

6) $\Longrightarrow$ 1) a) By theorem 1.1 and the corollary, it is clear that for any x in X, A(1) ≥ A(x).

b) Let x, y ∈ X and t = $\min(A(x*y), A(y))$. Then $x_t * y_t$ = $(x*y)_t \in \tilde{A}$ and $y_t \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, $x_t \in \tilde{A}$. So A(x) ≥ t = $\min(A(x*y), A(y))$.

2.POSITIVE IMPLICATIVE WEAK IDEAL

Definition 2.1 A non empty subset I of X is called a positive implicative ideal if it satisfies:

a) $1 \in I$

b) for all $x, y, z \in X, (x*y)*z \in I$ and $y*z \in I$ implies $x*z \in I$.

Definition 2.2 A fuzzy subset A of a BE-algebra X is a fuzzy positive implicative ideal if and only if:

a) for any $x \in X, A(1) \geq A(x)$,

b) for any $x, y, z \in X, A(x*z) \geq \min\left(A((x*y)*z), A(y*z)\right)$.

Definition 2.3 $\tilde{A}$ is a positive implicative weak ideal of $\tilde{X}$ if and only if:

a) For any $v \in IM(A), 1_v \in \tilde{A}$,

b) for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ such that $\left(x_\lambda * y_\mu\right) * z_\alpha \in \tilde{A}$, and $y_\mu * z_\alpha \in \tilde{A}$, we have $(x*z)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

Theorem 2.1 If $\tilde{A}$ is a weak ideal, then the following conditions are equivalent:

1) A is a fuzzy positive implicative ideal.

2) for all $x_\lambda, y_\mu \in \tilde{X}$, $(x_\lambda * y_\mu) * y_\mu \in \tilde{A}$ implies $x_\lambda * y_\mu \in \tilde{A}$

3) for any $t \in (0,1]$ the t-level subset $A^t = \{x \in X | A(x) \geq t\}$ is a positive implicative ideal when $A^t \neq \emptyset$.

4) for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ implies $(x_\lambda * z_\alpha) * (y_\mu * z_\alpha) \in \tilde{A}$.

5) for any $x, y, z \in X$, $A((x * z) * (y * z)) \geq A((x * y) * z)$.

6) $\tilde{A}$ is a positive implicative weak ideal.

Lemma 2.1 Suppose that I is an ideal of a BE algebra X. Then the following conditions are equivalent

i) I is positive implicative.

ii) $(x * y) * y \in I$ implies $x * y \in I$.

iii) $(x * y) * z \in I$ implies $(x * z) * (y * z) \in I$.

Proof. 1) ⇒ 2) Let $x_\lambda, y_\mu \in \tilde{X}$ and $(x_\lambda * y_\mu) * y_\mu \in \tilde{A}$. since A is fuzzy positive implicative,

A(x∗y) ≥ min(A((x∗y)∗y), A(y∗y) ≥ min(A((x∗y)∗y), A(1)) ≥ min(λ, μ).

So $x_\lambda * y_\mu = (x * y)_{\min(\lambda,\mu)} \in \tilde{A}$.

2) ⇒ 3) a) Let x∈ A^t. Then A(x) ≥ t. since A is a fuzzy ideal, A(1) ≥A(x).

So A(1)≥ $A(x) \geq t, so\ 1 \in A^t$.

b) If (x∗y)∗ $y \in A^t$, then $(x_t * y_t) * y_t \in \tilde{A}$. From the hypothesis, we obtain $x_t * y_t \in \tilde{A}$. Hence $x * y \in A^t$ .By lemma 2.1, A^t is a positive implicative ideal.

3) ⇒ 4) Let $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and t = min (λ, μ, α). Then (x∗ $y) * z \in A^t$. Since A^t is a positive implicative ideal ,we apply lemma 2.1 and obtain (x∗ $z) * (y * z) \in A^t$. So ((x∗ $z) * (y * z))_t = (x_\lambda * z_\alpha) * (y_\mu * z_\alpha) \in \tilde{A}$

4) ⇒ 5) Let x, y, z ∈ X and t = A((x∗ $y) * z)$, $((x * y) * z)_t = (x_t * y_t) * z_t \in \tilde{A}$. Using the hypothesis, we obtain $(x_t * z_t) * (y_t * z_t) = ((x * y) * (y * z))_t \in \tilde{A}$. So A((x∗ z)∗ $(y * z)) \geq t =$ A((x∗ y)∗ z).

5) ⇒ 6) a) Let $\nu \in Im(A)$. It is clear that $1_\nu \in \tilde{A}$.

b) Let $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $y_\mu * z_\alpha \in \tilde{A}$. Then A((x∗ y) ∗z) ≥ min(λ, μ, α) and A(y∗ z) ≥ min(μ, α).

From the hypothesis and the fact that A is a fuzzy ideal, we obtain

A(x∗ $z) \geq \min(A((x * z) * (y * z)), A(y * z)) \geq \min(A((x * y) * z), A(y * z)) \geq \min(\min(\lambda, \mu, \alpha), \min(\mu, \alpha)) = \min(\lambda, \mu, \alpha)$.

So$(x * z)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

6) ⇒ 1) a) let x ∈ X, it is clear that A(1) ≥ A(x).

b) Let x, y, z ∈ X and A((x∗ $y) * z) = \beta$, $A(y * z) = \alpha$. $((x * y) * z)_{\min(\beta,\alpha)} = (x_\beta * y_\alpha) * z_\alpha \in \tilde{A}$ and $y_\alpha * z_\alpha = (y * z)_\alpha \in \tilde{A}$. Since $\tilde{A}$ is a positive implicative weak ideal, we have $(x * z)_{\min(\beta,\alpha)} \in \tilde{A}$.

Hence A(x∗ z) ≥ min(β, α) = min(A((x∗ $y) * z), A(y * z)$).

3. COMMUTATIVE WEAK IDEAL

Definition 3.1

A non empty subset I of X is called a commutative ideal if it satisfies

a) 1∈ I,
b) (x∗y)∗z ∈ I and z ∈ I imply x∗ $(y * (y * x)) \in I$.

Definition 3.2

A fuzzy subset A of a BE – algebra X is a fuzzy commutative ideal if and only if :

a) for any x ∈ X, A(1) ≥ A(x),
b) for any x ,y, z ∈ X, A(x∗ $(y * (y * x))$) ≥ min$(A((x * y) * z)$, A(z)).

Definition 3.3

a) for any $\nu \in$ Im(A), $1_\nu \in \tilde{A}$,
b) for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ such that $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, we have $x_{\min(\lambda,\alpha)} * (y_\mu * (y_\mu * x_{\min(\lambda,\alpha)})) \in \tilde{A}$.

The following theorem give a characterization of a commutative weak ideal.

Theorem 3.1 Suppose that $\tilde{A}$ is a weak ideal, then the following conditions are equivalent:

1) A is commutative.
2) for all $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$, we have $x_{\min(\lambda,\mu)} * (y_\mu * (y_\mu * x_{\min(\lambda,\mu)})) \in \tilde{A}$.
3) for any t∈ (0,1] the t- level subset $A^t = \{x \in X | A(x) \geq t\}$ is a commutative ideal when $A^t \neq \emptyset$.
4) $\tilde{A}$ is commutative.

The proof is similar to theorem2.1 and is omitted.

4. IMPLICATIVE WEAK IDEAL

Definition 4.1

A non empty subset I of X is called an implicative ideal if it satisfies: for all x, y, z ∈ X,

a) 1 ∈ X,
b) [x∗ $(y * x)] * z \in$ I and $z \in$ I imply $x \in$ I.

Definition 4.2

A fuzzy subset A of a BE algebra X is a fuzzy implicative ideal if and only if :

a) for any x ∈ X, A(1) ≥ A(x),
b) for any x, y, z ∈ X, A(x) ≥ min$((A(x * (y * x)) * z)$, A(z)).

Definition 4.3

$\tilde{A}$ is an implicative weak ideal of $\tilde{X}$ if and only if:

a) for any $\nu \in Im(A), 1_\nu \in \tilde{A}$.

b) for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, $imply\ x_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

Theorem 4.1 Let A be a fuzzy subset of a BE algebra X, then the following conditions are equivalent:
1) A is a fuzzy implicative ideal.

2) $\tilde{A}$is an implicative weak ideal.

Proof. 1) ⇒ 2) a) Let $\lambda \in Im\ (A)$. Suppose that λ = A(x). since A is a fuzzy implicative ideal,

We have A(1) ≥ A(x) = λ. So $1_\lambda \in \tilde{A}$.

b)Let $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Then A((x$*$ $(y * x)) * z)$ $\geq \min(\lambda, \mu, \alpha)$ and A(z) ≥ α.
Since A is a fuzzy implicative ideal, we have A(x) ≥ min$(A((x * (y * x)) * z)$, A(z)) ≥
min$(min(\lambda, \mu, \alpha), \alpha) = \min(\lambda, \mu, \alpha)$. So $x_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

2) ⇒ 1) a) Let x∈ X and λ = A(x), λ ∈ Im (A). since $\tilde{A}$ is an implicative weak ideal, we have $1_\lambda \in \tilde{A}$.
So A(1) ≥ λ = A(x).

b) If x, y, z ∈X, let A((x$*$ $(y * x)) * z) = \beta\ and\ A(z) = \alpha$. Then ((x $*$ $(y * x)) * z)_{\min(\beta,\alpha)}$ = $(x_\beta * (y_\beta * x_\beta) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ is an implicative weak ideal, we have $x_{\min(\beta,\alpha)} \in \tilde{A}$. So A(x) ≥ min$(\beta, \alpha)$ = min(A((x$*$ $(y * x)) * z$, A(z)).

Theorem 4.2 Suppose that $\tilde{A}$ is a weak ideal then the following conditions are equivalent :

1) A is a fuzzy implicative ideal
2) for all $x_\lambda, y_\mu \in \tilde{X}$, $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$ implies $x_{\min(\lambda,\mu)} \in \tilde{A}$.
3) for t ∈ (0,1], the t-level subset A^t = {x ∈ X| A(x) ≥t} is an implicative ideal when $A^t \neq \emptyset$.
4) $\tilde{A}$ is implicative.

Before proving the theorem, we recall the following result:

Lemma4.1 An ideal I of a BE-algebra X is implicative if and only if for any x, y, z ∈ X such that x$*$ $(y * x) \in I$, we have x ∈ I.

Proof. 1)⇒ 2) Let $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$. Since A is fuzzy implicative, we have

A(x) ≥ min(A((x$*$ $(y * x) * 1), A(1)) \geq \min(\lambda, \mu)$. So $x_{\min(\lambda,\mu)} \in \tilde{A}$.

2) ⇒ 3) a) It is clear that 1 ∈ A^t.

b) Let x$*$ $(y * x) \in A^t$. Then $(x * (y * x))_t = x_t * (y_t * x_t) \in \tilde{A}$. Using the hypothesis, we obtain $x_t \in \tilde{A}$.
So x ∈ A^t and by lemma 4.1, A^t is implicative.

3) ⟹ 4) a) Let λ ∈ Im (A), it is clear that $1_\lambda \in \tilde{A}$.

b) If $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, let t = min(λ, μ, α). Then(x$*$ $(y * x)) * z \in A^t$ and $z \in A^t$.
Because A^t is implicative, x ∈ A^t. So $x_{\min(\lambda,\mu,\alpha)} = x_t \in \tilde{A}$.

4) ⟹ 1) Following from theorem 4.1.

The following theorem gives some equivalent conditions for a subalgebra $\tilde{A}$ to be an implicative weak ideal.

Theorem 4.3 If $\tilde{A}$ is a subalgebra (namely A is a fuzzy subalgebra by theorem 1.1), the following conditions are equivalent:

1) A is a fuzzy implicative ideal.

2) For any $x_\lambda, y_\mu, z_\alpha, w_t \in \tilde{X}$, $\left(\left(x_\lambda * \left(y_\mu * x_\lambda\right)\right) * z_\alpha\right) * w_t = 1_{\min(\lambda,\mu,\alpha,t)}$ with z_α and $w_t \in \tilde{A}$ imply $x_{\min(\lambda,\mu,\alpha,t)} \in \tilde{A}$.
3) Or any x, y, z, w ∈ X, ((x$*$ $(y * x)) * z) * w$ = 1 imply A(x) ≥ min(A(z), A(w)).
4) $\tilde{A}$ is an implicative weak ideal.

Proof. 1)⇒ 2) Let $\left(\left(x_\lambda * \left(y_\mu * x_\lambda\right)\right) * z_\alpha\right) * w_t = 1_{\min(\lambda,\mu,\alpha,t)}$ with z_α and $w_t \in \tilde{A}$. since A is fuzzy implicative ideal, A is also a fuzzy ideal, we have $x_{\min(\lambda,\alpha,t)} * (y_\mu * x_{\min(\lambda,\alpha,t)}) = x * (y * x)_{\min(\lambda,\mu,\alpha,t)} \in \tilde{A}$. Using Theorem 4.2, we have $x_{\min(\lambda,\mu,\alpha,t)} \in \tilde{A}$.

2) ⇒ 3) Let ((x$*$ $(y * x)) * z) * w$ = 1 and t = min(A(z), A(w)), then z_t and $w_t \in \tilde{A}$

$((x_t * (y_t * x_t)) * z_{t)} w_t = 1_t$. Using the hypothesis, we obtain $x_t \in \tilde{A}$. So $A(x) \geq t = \min(A(z), A(w))$

3)⟹ 4) a) Let $\lambda \in Im(A)$, because $\tilde{A}$ is a subalgebra, we have $1_\lambda \in \tilde{A}$ (by lemma 1.1).

b) Let $\left(x_\lambda * \left(y_\mu * x_\lambda\right)\right) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since {(x$*$ $(y * x)) * [($x $* (y * x)) * z]\} * \quad z = 1$, we apply the hypothesis and obtain $A(x) \geq \min(A(($x $* (y * x) * z),\ A(z) \geq \min(\min(\lambda, \mu, \alpha), \alpha) = \min(\lambda, \mu, \alpha)$. So $x_{(\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

4)⟹1) Follows from Theorem 4.1.

Now we describe the relation between positive implicative weak ideal and implicative weak ideal.

Theorem 4.4 If $\widetilde{A}$ is an implicative weak ideal, then $\widetilde{A}$ is a positive implicative weak ideal.

Proof. Suppose that $\widetilde{A}$ is an implicative weak ideal of $\widetilde{X}$ and let $x_\lambda, y_\mu, z_\alpha \in \widetilde{X}$ such that $\left(x_\lambda * y_\mu\right) * z_\alpha \in \widetilde{A}$ and $y_\mu * z_\alpha \in \widetilde{A}$. To Prove that $\widetilde{A}$ is a positive implicative, we need only to show that $(x * z)_{\min(\lambda,\mu,\alpha)} \in \widetilde{A}$. Using (9′) , $\{[(x_\lambda * z_\alpha) * z_\alpha] * \left(y_\mu * z_\alpha\right)\} * [(x_\lambda * z_\alpha) * y_\mu] = 1_{\min(\lambda,\mu,\alpha)}$. By (5′) $(x_\lambda * z_\alpha) * y_\mu = (x_\lambda * y_m u) * z_\alpha$. since $\widetilde{A}$ is a weak ideal, we obtain $(x_{\min(\lambda,\mu)} * z_\alpha) * z_\alpha \in \widetilde{A}$. from(5′) we obtain $\left(x_{\min(\lambda,\mu)} * z_\alpha\right) * \left[x_{\min(\lambda,\mu)} * \left(x_{\min(\lambda,\mu)} * z_\alpha\right)\right] = \left\{x_{\min(\lambda,\mu)} * \left[x_{\min(\lambda,\mu)} * \left(x_{\min(\lambda,\mu)} * z_\alpha\right)\right]\right\} * z_\alpha = [x_{\min(\lambda,\mu)} * z_\alpha] * z_\alpha$ from (7′) It follows that $\left\{\left(x_{\min(\lambda,\mu)} * z_\alpha\right) * \left[x_{\min(\lambda,\mu)} * \left(x_{\min(\lambda,\mu)} * z_\alpha\right)\right]\right] * 1_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$. Since $\tilde{A}$ is an implicative weak ideal, we obtain $x_{\min(\lambda,\mu)} * z_\alpha = (x * z)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$ which completes the proof.

Corollary If A is a fuzzy implicative ideal, then A is a positive implicative ideal.

For further relations between a positive implicative weak ideal and an implicative weak ideal, we have the following:

Theorem 4.5 Let $\tilde{A}$ be a positive implicative weak ideal, then $\tilde{A}$ is a implicative if and only if for any $x_\lambda, y_\mu \in \tilde{X}$ such that $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$ we have $x_\lambda * (x_\lambda * y_\mu) \in \tilde{A}$.

Proof. Suppose that $\tilde{A}$ is an implicative weak ideal of $\tilde{X}$ and $y_\mu * \left(y_\mu * x_\lambda\right) \in \tilde{A}$. We want to show that $x_\lambda * (x_\lambda * y_\mu) \in \tilde{A}$. By (6′) $\left[x_\lambda * \left(x_\lambda * y_\mu\right)\right] * x_\lambda = 1_{\min(\lambda,\mu)}$. By (4′), $\left(y_\mu * x_\lambda\right) * \left\{y_\mu * \left[x_\lambda * \left(x_\lambda * y_\mu\right)\right]\right\} = 1_{\min(\lambda,\mu)}$. By(4′) $\{\left[x_\lambda * \left(x_\lambda * y_\mu\right)\right] * \left\{y_\mu * \left[x_\lambda * \left(x_\lambda * y_\mu\right)\right]\right\}\} * \{ [x_\lambda * \left(x_\lambda * y_\mu\right)] * \left(y_\mu * x_\lambda\right)\} = 1_{\min(\lambda,\mu)}$. By (5′), $\left[x_\lambda * \left(x_\lambda * y_\mu\right)\right] * \left(y_\mu * x_\lambda\right) = \left[x_\lambda * \left(y_\mu * x_\lambda\right)\right] * \left(x_\lambda * y_\mu\right)$. by BE (2′) $\{\left[x_\lambda * \left(y_\mu * x_\lambda\right)\right] * \left(x_\lambda * y_\mu\right)\} * \left\{y_\mu * \left[x_\lambda * \left(x_\lambda * y_\mu\right)\right]\right\} = 1_{\min(\lambda,\mu)}$. From the fact that $\tilde{A}$ is a weak ideal we obtain

$\{[x_\lambda * (x_\lambda * y_\mu)] * \{y_\mu * [x_\lambda * (x_\lambda * y_\mu)]\}\} * 1_{\min(\lambda,\mu)} \in \tilde{A}$. Since $\tilde{A}$ is an implicative weak ideal, we have $x_\lambda * (x_\lambda * y_\mu) \in \tilde{A}$.

Conversely, if $[x_\lambda * (y_\mu * x_\lambda)] * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, we obtain $x_{\min(\lambda,\alpha)} * (y_\mu * x_{\min(\lambda,\alpha)}) \in \tilde{A}$. Let $\beta = \min(\lambda,\alpha)$ and $x_\beta * (y_\mu * x_\beta) \in \tilde{A}$. By BE (2′) $\{[y_\mu * (y_\mu * x_\beta)] * (y_\mu * x_\beta)\} * [x_\beta * (y_\mu * x_\beta)] = 1_{\min(\beta,\mu)}$. Using the fact that $\tilde{A}$ is a weak ideal, we obtain $[y_\mu * (y_\mu * x_\beta)] * (y_\mu * x_\beta) \in \tilde{A}$. Since $\tilde{A}$ is a positive implicative, we have $y_\mu * (y_\mu * x_\beta) \in \tilde{A}$. By applying the hypothesis, we obtain $x_\beta * (x_\beta * y_\mu) \in \tilde{A}$. **from BE (2′) $\{(x_\beta * y_\mu) * [x_\beta * (y_\mu * x_\beta)]\} * [(y_\mu * x_\beta) * y_\mu] = 1_{\min(\beta,\mu)}$. Also $(y_\mu * x_\beta) * y_\mu = 1_{\min(\beta,\mu)}$. So $(x_\beta * y_\mu) * [x_\beta * (y_\mu * x_\beta)] = 1_{\min(\beta,\mu)}$. Using the fact that $\tilde{A}$ is a weak ideal, we obtain $(x_\beta * y_\mu) \in \tilde{A}$. Because $x_\beta * (x_\beta * y_\mu) \in \tilde{A}$ (see **). We have $x_{\min(\beta,\mu)} = x_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

Theorem4.6 A weak ideal $\tilde{A}$ is implicative if and only if it is both commutative and implicative.

Proof. Suppose that $\tilde{A}$ is an implicative weak ideal, from Theorem 4.4, we know $\tilde{A}$ is a positive implicative weak ideal. To prove that $\tilde{A}$ is a commutative weak ideal, we need only to show that $\tilde{A}$ satisfies the condition 2 of Theorem 3.1. Let $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$. By(6′) $\{x_\lambda * [y_\mu * (y_\mu * x_\lambda)]\} * x_\lambda = 1_{\min(\lambda,\mu)}$. By(3′) $(y_\mu * x_\lambda) * \{y_\mu * [x_\lambda * [y_\mu * (y_\mu * x_\lambda)]]\} = 1_{\min(\lambda,\mu)}$. Put $t_\beta = x_\lambda * [y_\mu * (y_\mu * x_\lambda)]$. We have $(y_\mu * x_\lambda) * (y_\mu * t_\beta) = 1_{\min(\lambda,\beta,\mu)}$. Applying(4′) we obtain $[t_\beta * (y_\mu * t_\beta)] * [t_\beta * (y_\mu * x_\lambda)] = 1_{\min(\lambda,\beta,\mu)}$. But $t_\beta * (y_\mu * X_\lambda) = \{x_\lambda * [y_\mu * (y_\mu * x_\lambda)]\} * (y_\mu * x_\lambda) = [x_\lambda * (y_\mu * x_\lambda)] * [y_\mu * (y_\mu * x_\lambda)]$ by(5′). From (9′) we also have $[[x_\lambda * (y_\mu * x_\lambda)] * [y_\mu * (y_\mu * x_\lambda)]] * (x_\lambda * y_\mu) = 1_{\min(\lambda,\mu)}$. Since $x_\lambda * y_\mu \in \tilde{A}$. , we obtain $t_\beta * (y_\mu * x_\lambda) \in \tilde{A}$. so $t_\beta * (y_\mu * t_\beta) \in \tilde{A}$. $\tilde{A}$ is an implicative weak ideal. Hence we applying theorem 4.3 and obtain $t_\beta = t_{\min(\beta,\mu)} \in \tilde{A}$. So $t_\beta = X_\lambda * (y_\mu * (y_\mu * x_\lambda)) = x_{\min(\lambda,\mu)} * [y_\mu[y_\mu * x_{\min(\lambda,\mu)}]] \in \tilde{A}$.

Conversely, suppose that $\tilde{A}$ is both commutative and positive implicative, we must verify that $\tilde{A}$ is implicative. Using Theorem 4.3, we need only to show that $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$ implies $x_{\min(\lambda,\mu)} \in \tilde{A}$. By BE (2′)$[y_\mu * (y_\mu * x_\lambda)] * x_\lambda = 1_{\min(\lambda,\mu)}$. By (4′) $\{[y_\mu * (y_\mu * x_\lambda)] * (y_\mu * x_\lambda)\} * [x_\lambda * (y_\mu * x_\lambda)] = 1_{\min(\lambda,\mu)}$. Since $\tilde{A}$ is a weak ideal and $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$, we obtain $[y_\mu * (y_\mu * x_\lambda)] * (y_\mu * x_\lambda) \in \tilde{A}$. Using Theorem 2.1, we obtain $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$. On the other hand using BE (2′) we have $\{(x_\lambda * y_\mu) * [x_\lambda * (y_\mu * x_\lambda)]\} * [(y_\mu * x_\lambda) * y_\mu] = 1_{\min(\lambda,\mu)}$. Since $(y_\mu * x_\lambda) * y_\mu = 1_{\min(\lambda,\mu)}$, we

have $(x_\lambda * y_\mu) * [x_\lambda * (y_\mu * x_\lambda)] = 1_{\min(\lambda,\mu)}$. From $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$, we obtain $x_\lambda * y_\mu \in \tilde{A}$. Since $\tilde{A}$ is a commutative, we apply the theorem 3.1 and obtain $x_\lambda * [y_\mu * (y_\mu * x_\lambda)] \in \tilde{A}$. Since $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$, we obtain $x_{\min(\lambda,\mu)} \in \tilde{A}$.

CONCLUSION

This paper shows how the concept of fuzzy ideals and weak ideals can be used with BE algebras and Solving theorems. The proposed approach can be extended to Weak ideal, Positive implicative weak ideal, Commutative weak ideal and Implicative weak ideal.

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