On Addition and Multiplication of Double Representation of q-k-normal Matrices

R.Kavitha, K.Gunasekaran
Lecturer, Head Of the Department
Ramanujan Research Centre, PG and Research Department of Mathematics,
Government Arts College (Autonomous), Kumbakonam – 612 002, Tamil Nadu, India.

Abstract: In this paper, some results of addition and multiplication of q-𝓀-normal matrices through double representation $A = A_0 + A_1 j \in H_{n\times n}$, where A_0 and A_1 are complex matrices of same order of A.

AMS Classifications : 15A09, 15A57, 15A24, 15A33, 15A15

Keywords : Quaternion-𝓀-normal matrices, q-𝓀-unitary matrix, q-𝓀-Hermitian matrix, double representation matrix

1. Introduction

Wu.J.L and Zhang.P introduced bicomplex representation method for quaternion matrix is 2011[5]. The multiplication $A * B$ was defined as $A_0B_0 + A_1B_1 j$ where $A_0 + A_1 j = A$, $B_0 + B_1 j = B$, A_0, B_0, A_1, B_1 are complex matrices, in the study of quaternion division algebra. People always except to get some relation between quaternion division algebra and real algebra or complex algebra. However, some conclusions on real or complex fields but not on quaternion division algebra. It makes has to establish in matrix theory.

The addition and multiplication are q-𝓀-normal matrices by using double complex representation of quaternion matrices, the properties and conditions for addition and multiplication are given.

2. Some Definitions and Theorems

Theorem 2.1

If A and B are q-𝓀-normal matrices with having $A = A_0 + A_1 j$ and $B = B_0 + B_1 j$, $A_0, A_1, B_0, B_1 \in C_{n\times n}$ then $A + B$ is q-𝓀-normal when $A_0 + B_0$ and $A_1 + B_1$ are 𝓀-normal.

Proof

Since A and B are q-𝓀-normal.

So, we can write $AKA^*K = KA^*KA$ and $BKB^*K = KB^*KB$.

By definition of double representation $A = A_0 + A_1 j$ implies that $KA^*K = KA_0^*K - KA_1^*Kj$

Similarly $B = B_0 + B_1 j$ and $KB^*K = KB_0^*K - KB_1^*Kj$

Now, $(A + B) = (A_0 + B_0) + (A_1 + B_1) j$

So, $K(A + B)^* K = K(A_0 + B_0)^* K - K(A_1 + B_1)^* Kj$

$$(A + B)K(A + B)^* K = ((A_0 + B_0) + (A_1 + B_1)j)(K(A_0 + B_0)^* K) - K(A_1 + B_1)^* Kj)$$

Since $AB = A_0B_0 + A_1B_1j$

$$(A+B)K(A+B)^*K = (A_0+B_0)K(A_0+B_0)^*K - (A_1+B_1)K(A_1+B_1)^*Kj$$

Since A_0+B_0 and A_1+B_1 are k-normal

$$(A+B)K(A+B)^*K = K(A_0+B_0)^*K(A_0+B_0) - K(A_1+B_1)^*K(A_1+B_1)j$$

$$= (K(A_0+B_0)^*K - K(A_1+B_1)^*Kj)((A_0+B_0)+(A_1+B_1)j)$$

$$= K((A_0+B_0)+(A_1+B_1)j)^*K((A_0+B_0)+(A_1+B_1)j)$$

$$= K(A+B)^*K(A+B)$$

That is $(A+B)K(A+B)^*K = K(A+B)^*K(A+B)$

Thus $A+B$ is q-k-normal matrix.

Hence proved.

Theorem 2.2

Let A and B are q-k-normal in $H_{n\times n}$ and A_s B_s; $s = 0,1$ are k-normal in $C_{n\times n}$ then AB is q-k-normal when $A_sKB_s{}^*K = KB_s{}^*KA_s$ and $KA_s{}^*KB_s = B_sKA_s{}^*K$.

Proof

Since A and B are q-k-normal $AKA^*K = KA^*KA$ and $BKB^*K = KB^*KB$. Since A_s and B_s are k-normal. So $A_sKA_s{}^*K = KA_s{}^*KA$ and $B_sKB_s{}^*K = KB_s{}^*KB_s$.

Now,

$$(AB)(K(AB)^*K) = (A_0B_0 + A_1B_1j)(K(A_0B_0)^*K - K(A_1B_1)^*Kj) \quad [\text{Since } AB = A_0B_0 + A_1B_1j]$$

$$= (A_0B_0)K(A_0B_0)^*K - (A_1B_1)K(A_1B_1)^*Kj$$

$$= A_0B_0KB_0{}^*A_0{}^*K - A_1B_1KB_1{}^*A_1{}^*Kj$$

$$= A_0B_0KB_0{}^*KKA_0{}^*K - A_1B_1KB_1{}^*KKA_1{}^*Kj \quad [\text{since } K^2 = I]$$

$$= A_0KB_0{}^*KB_0KA_0{}^*K - A_1KB_1{}^*KB_1KA_1{}^*Kj \quad [\text{since } B_0, B_1 \text{ are } k\text{-normal}]$$

Since $A_sKB_s{}^*K = KB_s{}^*KA_s$, $KA_s{}^*KB_s = B_sKA_s{}^*K$

$$(AB)(K(AB)^*K) = KB_0{}^*KA_0KA_0{}^*KB_0 - KB_1{}^*KA_1KA_1{}^*KB_1j$$

$$= (KB_0{}^*K)(KA_0{}^*K)A_0B_0 - (KB_1{}^*K)(KA_1{}^*K)A_1B_1j \quad [\text{Since } A_0, A_1 \text{ are } k\text{-normal}]$$

$$= K(A_0B_0)^*K(A_0B_0) - K(A_1B_1)^*K(A_1B_1)j$$

$$= K((A_0B_0)^* - (A_1B_1)^*j)K(A_0B_0 + A_1B_1j)$$

$$= K(AB)^*K(AB)$$

Thus AB is q-$\boldsymbol{k}$-normal

Hence proved

Definition 2.3

A double representation matrix $A \in H_{n\times n}$ is said to be q-$\boldsymbol{k}$-unitary if $A_0KA_0^{*}K - A_1KA_1^{*}Kj = I$.

Remark 2.4

If $A_0KA_0^{*}K = I + A_1KA_1^{*}Kj$ and $A_1KA_1^{*}Kj = A_0KA_0^{*}K - I$ are implied by the definition (2.3).

Remark 2.5

Let A,B be double representation of complex matrices [3] there exist an q-$\boldsymbol{k}$-unitary matrix $U = U_0 + U_1 j$ such that $B = (KU^{*}K)AU$.

From this, we have

$$B_0 + B_1 j = (KU_0^{*}K - KU_1^{*}Kj)(A_0 + A_1 j)(U_0 + U_1 j)$$

$$= (KU_0^{*}K)A_0U_0 - (KU_1^{*}K)A_1U_1 j$$

Equating left hand and right hand sides $B_0 = (KU_0^{*}K)A_0U_0$ and $B_1 = -(KU_1^{*}K)A_1U_1$

Example 2.6

$A = \begin{pmatrix} 1+i & 2i \\ 3+2i & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2+2i & 2+3i \\ -2+2i & -3+2i \end{pmatrix}$ if we take $U = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$ are simply verified that $B_0 = (KU_0^{*}K)A_0U_0$ and $B_1 = -(KU_1^{*}K)A_1U_1$.

Theorem 2.7

Let $A = A_0 + A_1 j \in H_{n\times n}$. If A is q-$\boldsymbol{k}$-unitarily equivalent to a diagonal matrix $D = D_0 + D_1 j$ where $D_0, D_1 \in C_{n\times n}$, then A_0 and A_1 are $\boldsymbol{k}$-normal.

Proof

Since $A = A_0 + A_1 j \in H_{n\times n}$. If we assume that A is q-$\boldsymbol{k}$-unitarily equivalent to a diagonal $D = D_0 + D_1 j$.

Therefore, there exist an q-$\boldsymbol{k}$-unitary matrix $P = P_0 + P_1 j$ such that $(KP^{*}K)AP = D$.

That is $K(P_0 + P_1 j)^{*}K(A_0 + A_1 j)(P_0 + P_1 j) = (D_0 + D_1 j)$

$$\Rightarrow (KP_0^{*}K)A_0P_0 - (KP_1^{*}K)A_1P_1 j = D_0 + D_1 j$$

This equation is pre and post multiplied by P and $KP^{*}K$ on both sides respectively.

We have, $P_0(KP_0^{*}K)A_0P_0(KP_0^{*}K) + P_1(KP_1^{*}K)A_1P_1(KP_1^{*}K)j = P_0D_0(KP_0^{*}K) - P_1D_1(KP_1^{*}K)j$

Equating the component wise.

We have $P_0(KP_0^{*}K)A_0P_0(KP_0^{*}K) = P_0D_0(KP_0^{*}K)$ and

$$P_1(KP_1^*K)A_1P_1(KP_1^*K) = -P_1D_1(KP_1^*K)$$

This implies that

$$A_0 = P_0D_0(KP_0^*K) \text{ as } P_0KP_0^*K = I$$

$$A_0^* = [P_0D_0(KP_0^*K)]^*$$

$$= KP_0KD_0^*P_0^*$$

Now, $A_0(KA_0^*K) = (P_0D_0KP_0^*K)K(KP_0KD_0^*P_0^*)K$

$$= P_0D_0KP_0^*KP_0KD_0^*P_0^*K \quad [\text{since } K^2 = I]$$

$$= P_0D_0(KP_0^*KP_0)(KD_0^*P_0^*K)$$

$$= P_0D_0KD_0^*P_0^*K \quad [\text{since } (KP_0^*K)P_0) = I]$$

$$= P_0D_0(KD_0^*K)(KP_0^*K) \quad [\text{since } K^2 = I]$$

$$(KA_0^*K)A_0 = K(KP_0KD_0^*P_0^*)K(P_0D_0KP_0^*K)$$

$$= P_0KD_0^*P_0^*KP_0D_0KP_0^*K$$

$$= P_0(KD_0^*K)(KP_0^*K)P_0D_0(KP_0^*K) \quad [\text{since } K^2 = I]$$

$$= P_0(KD_0^*K)(KP_0^*KP_0)D_0(KP_0^*K)$$

$$= P_0(KD_0^*K)D_0(KP_0^*K) \quad [\text{since } KP_0^*KP_0 = I]$$

Therefore, D_0 and KD_0^*K are each diagonal. So, $D_0(KD_0^*K) = (KD_0^*K)D_0$ and hence $A_0(KA_0^*K) = (KA_0^*K)A_0$.

So, A_0 is *k*-normal.

Similarly we may prove that A_1 is *k*-normal.

Remark 2.8

From the above theorem (2.7) we can prove that A is q-*k*-normal by using the theorem

That is A_0 and A_1 are *k*-normal then $A = A_0 + A_1 j$ is *k*-normal.

Theorem 2.9

If A is q-*k*-Hermitian and A_0 and A_1 are also *k*-Hermitian then $A^{-1}(KA^*K) = -4I$.

Proof

Since $A = A_0 + A_1 j$ So $A^* = A_0^* - A_1^* j$ and $A^{-1} = A_0^{-1} - A_1^{-1} j$.

Now, $A^{-1}(KA^*K)K(A^{-1}KA^*K)^*K$

$$= (A_0^{-1} - A_1^{-1}j)(KA_0^*K - KA_1^*Kj)K[(A_0^{-1} - A_1^{-1}j)(KA_0^*K - KA_1^*Kj)]^*K$$

$$= (A_0^{-1}KA_0^*K + A_1^{-1}KA_1^*Kj)K[(A_0^{-1} - A_1^{-1}j)(A_0 - A_1 j)]^*K$$

Since A_0 and A_1 *k*-Hermitian

$= (A_0^{-1}KA_0^{*}K + A_1^{-1}KA_1^{*}Kj)K(A_0^{-1}A_0 + A_1^{-1}A_1 j)^{*}K$

$= (A_0^{-1}KA_0^{*}K + A_1^{-1}KA_1^{*}Kj)K2I^{*}jK$ [Since $A_0A_0^{-1} = I$]

$= 2(A_0^{-1}KA_0^{*}K + A_1^{-1}KA_1^{*}Kj)jI$

$= 2(A_0^{-1}A_0 + A_1^{-1}A_1 j)jI$

$= 2\times 2(j.j)$

$= -4I$

Hence proved.

Remark 2.10

From the theorem (2.9) we can get the another theorem that if A is q-𝓀-normal, A_0 and A_1 are 𝓀-normal then $A^{-1}[KA^{*}K] = -4I$.

Remark 2.11

We combine theorem (2.9) and the remark (2.10) , whether the A is q-𝓀-Hermitian or q-𝓀-normal the $A^{-1}KA^{*}K$ is identity with multiple of -4.

References

1. Bhatia, Rajendara: Matrix Analysis; Springer Publications(1997) 159 – 164
2. Chen.L.X, "Inverse Matrix and Properties of Double Determinant over Quaternion TH Field," Science in China (Series A), Vol. 34, No. 5, 1991, pp. 25-35.
3. Gunasekaran.K and Kavitha.R: On Quaternion-𝓀-normal matrices; International Journal of Mathematical Archive-7(7),(2016) 93-101.
4. Goyal.S.P and Ritu.G, "The Bicomplex Hurwitz Zeta function," The South East Asian Journal of Mathematics and Mathematical Sciences, 2006.
5. Li.T.S, "Properties of Double Determinant over Quaternion Field," Journal of Central China Normal University, Vol. 1, 1995, 3-7.
6. Rochon.D, "A Bicomplex Riemann Zeta Function," Tokyo Journal of Mathematics, Vol. 27, No. 2, 2004, pp.357-369.
7. Wu.J.L nd Zhang.P " On Bicomplex Representation Methods and Application of Matrices over Quaternionic Division Algebra", Ad.in Pure Maths. Vol.1 pp. 9-15.
8. Zhang.Q.C, "Properties of Double Determinant over the Quaternion Field and Its Applications," Acta MathematicaSinica, Vol. 38, No. 2, 1995, pp. 253-259.
9. Zhang.F, Quarternions and Matrices of quaternions, linear Algebra and its application, 251 (1997), 21 -57.
10. Zhuang.W.J, "Inequalities of Eigenvalues and Singular Values for Quaternion Matrices," Advances in Mathematics, Vol. 4, 1988, pp. 403-406.