Prime Pair Labeling Of Directed Graphs

[1] K.Palani & [2]G.Suganya
[1]PG & Research Department of Mathematics
[2]Research Scholar
A.P.C. Mahalaxmi College for Women, Thoothukudi
Manonmanium Sundaranar University, Tirunelveli, TN, India

ABSTRACT:

Let D(p,q) be a digraph. A function f:V→{1,2,…,p+q} is said to be a prime pair labeling of D if it is both an in and outdegree prime pair labeling of D. In this paper, we introduce some digraphs with particular names depending upon the orientation of the corresponding simple graphs. Further, we develop a new concept prime pair labeling in digraphs. Also, investigate the existence of the same in the digraph introduced previously.

Keywords: prime pair labeling.

AMS Subject Classification: 05C78

1. INTRODUCTION:

A directed graph or digraph D consists of a finite set V of vertices (points) and a collection of ordered pairs of distinct vertices. Any such pair (u,v) is called an arc or directed line and will usually be denoted by $\overrightarrow{uv}$. The arc $\overrightarrow{uv}$ goes from u to v and incident with u and v, we also say u is adjacent to v and v is adjacent from u. A digraph D with p vertices and q arcs is denoted by D (p, q). The indegree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The outdegree $d^+(v)$ of v is the number of arcs having v as its initial vertex [1].A labeling of a graph G is an assignment of integers to either the vertices or the edges or both subject to certain conditions. The notion of prime labeling for graphs originated with Roger Entringer and was introduced in the paper by Tout et.al[7] in the early 1980's and since then it is an active field of research for many scholars. In & Outdegree prime pair labeling in graphs were introduced by K. palani et.al[5,6]. Let D(p,q) be a digraph. A function f:V→{1,2,…,p+q} is said to be an indegree prime pair labeling of D if at each $u \in V(D)$, gcd$[f(v), f(w)] = 1, \forall\, v, w \in N^-(u)$, where $N^-(u) = \{w \in V(D) | wu \in A(D)\}$. A function f:V→{1,2,…,p+q} is said to be an outdegree prime pair labeling of D, if at each $u \in V(D)$, gcd$[f(v), f(w)] = 1, \forall\, v, w \in N^+(u)$, where $N^+(u) = \{w \in V(D) | uw \in A(D)\}$. Also, it is proved that digraphs which admit indegree prime pair labeling need not admit outdegree prime pair labeling and vice versa. Further, even if a digraph admits both in & outdegree prime pair labelings then both need not be the same. This fact motivated us to define the new concept prime pair labeling in digraphs. A prime pair labeling in digraph 'D' is a fn f:V→{1,2,…,p+q} which is both an indegree prime pair labeling and an outdegree prime pair labeling. A star graph $K_{1,n}$ [3] is a tree with one internal node and n-leaves (but no internal node and n+1leaves when n ≤ 1). A wheel graph W_n [3] is a graph that contains a cycle of order n−1 and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. The wheel W_n can be defined as the graph K_1+C_{n-1} where K_1 is the singleton graph and C_{n-1} is the cycle graph. The comb $P_n \odot K_1$ [3] is obtained by joining a pendent edge to each vertex of P_n. The crown $C_n \odot K_1$ [3] is obtained by joining a pendent edge to each vertex of C_n. A dragon D_nm [4] is a graph obtained by joining the end point of path P_m to the cycle C_n. In this paper, we introduce some digraphs with particular names depending upon the orientation of the corresponding simple graphs. Further, we develop a new concept prime pair labeling in digraphs. Also, investigate the existance of the same in the digraph introduced previously. The following fact is useful in assigning the prime pair Labeling. "Gcd of any two consecutive integer is 1".

1.1 Theorem[2]: Bertrand-chebyshev theorem: $\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1$, for all x ≥ 2 where $\pi(x)$ is the prime counting function (number of primes less than or equal to x).

2. DIGRAPHS FROM SOME STANDARD AND SPECIAL GRAPHS

In this section we define and specify some digraphs from the standard and special graphs.

2.1 Directed Path ($\overrightarrow{P_n}$): A Path P_n in which all the edges are directed in one direction is called a directed path and is denoted as $\overrightarrow{P_n}$.

2.2 Alternating path (A$\overrightarrow{P_n}$): A Path P_n in which the edges are given alternative direction is called an alternating path and is denoted as A$\overrightarrow{P_n}$.

2.3 Directed cycle ($\overrightarrow{C_n}$): A cycle C_n in which all the edges are directed clockwise or anticlockwise is called a directed cycle and is denoted as $\overrightarrow{C_n}$.

2.4 Alternating cycle (A$\overrightarrow{C_n}$): A cycle C_n with n even in which the edges are given alternative direction is called an alternating cycle and is denoted as A$\overrightarrow{C_n}$.

2.5 Instar (i$\overrightarrow{K_{1,n}}$): A Star graph $K_{1,n}$ in which all the edges are directed towards the root vertex is called an instar and is denoted as i$\overrightarrow{K_{1,n}}$.

2.6 Outstar (o$\overrightarrow{K_{1,n}}$): A Star graph $K_{1,n}$ in which all the edges are directed away from the root vertex is called an outstar and is denoted as o$\overrightarrow{K_{1,n}}$.

2.7 Alternating star (A$\overrightarrow{K_{1,n}}$): A Star graph $K_{1,n}$ with n even in which the edges are oriented alternatively is called an alternating star and is denoted as A$\overrightarrow{K_{1,n}}$.

2.8 Inwheel (i$\overrightarrow{W_n}$): A wheel graph W_n in which the edges of the outer cycle are directed clockwise or anticlockwise and the spoke edges are directed towards the central (or Hub) vertex is called an inwheel and is denoted as i$\overrightarrow{W_n}$.

2.9 Outwheel (o$\overrightarrow{W_n}$): A wheel graph W_n in which the edges of the outer cycle are directed clockwise or anticlockwise and the spoke edges are directed away from the central (or Hub) vertex is called an outwheel and is denoted as o$\overrightarrow{W_n}$.

2.10 Alternating wheel (A$\overrightarrow{W_n}$): A wheel graph W_n, with n odd is said to be an alternating wheel if the edges of the outer cycle are directed clockwise or anticlockwise and the spoke edges are oriented alternatiely and is denoted as A$\overrightarrow{W_n}$.

2.11 Alternating inwheel (Ai$\overrightarrow{W_n}$): A wheel graph W_n, with n odd is said to be an alternating inwheel if the edges of the outer cycle are oriented alternatively and the spoke edges are directed towards the central (or Hub) vertex and is denoted as Ai$\overrightarrow{W_n}$.

2.12 Alternating outwheel (Ao$\overrightarrow{W_n}$): A wheel graph W_n, with n odd is said to be an alternating outwheel if the edges of the outer cycle are oriented alternatively and the spoke edges are directed away from the central (or Hub) vertex and is denoted as Ao$\overrightarrow{W_n}$.

2.13 Double alternating wheel (DA$\overrightarrow{W_n}$): A wheel graph W_n, with n odd is said to be a double alternating wheel if both the edges of the outer cycle and the spoke edges are oriented alternatiely and is denoted as DA$\overrightarrow{W_n}$.

2.14 Upcomb (Up$\overrightarrow{P_n \odot K_1}$): A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented away from the end vertices is called an upcomb and is denoted as Up$\overrightarrow{P_n \odot K_1}$.

2.15 Downcomb (Down$\overrightarrow{P_n \odot K_1}$): A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented towards the end vertices is called a downcomb and is denoted as Down$\overrightarrow{P_n \odot K_1}$.

2.16 Alternating comb (A$\overrightarrow{P_n \odot K_1}$): A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented alternatively is called an alternating comb and is denoted as A$\overrightarrow{P_n \odot K_1}$.

2.17 Alternating upcomb (AUp$\overrightarrow{P_n \odot K_1}$): A comb graph $P_n \odot K_1$ in which the path edges are oriented alternatively and the pendent edges are oriented away from the end vertices is called an alternating upcomb and is denoted as AUp$\overrightarrow{P_n \odot K_1}$.

2.18 Alternating downcomb (ADown$\overrightarrow{P_n \odot K_1}$): A comb graph $P_n \odot K_1$ in which the path edges are oriented alternatively and the pendent edges are oriented towards the end vertices is called an alternating downcomb and is denoted as ADown$\overrightarrow{P_n \odot K_1}$.

2.19 Double alternating comb (DA$\overrightarrow{P_n \odot K_1}$): A comb graph $P_n \odot K_1$ in which both the path edges and the pendent edges are oriented alternatively is called a double alternating comb and is denoted as DA$\overrightarrow{P_n \odot K_1}$.

2.20 Incrown (i$\overrightarrow{C_n \odot K_1}$): A crown graph $C_n \odot K_1$ in which the edges of the cycle are directed clockwise or anticlockwise and the pendent edges are directed towards the cycle is called an incrown and is denoted as i$\overrightarrow{C_n \odot K_1}$.

2.21 Outcrown (o$\overrightarrow{C_n \odot K_1}$): A crown graph $C_n \odot K_1$ in which the edges of the cycle are directed clockwise or anticlockwise and the pendent edges are directed away from the cycle is called an outcrown and is denoted as o$\overrightarrow{C_n \odot K_1}$.

2.22 Alternating crown (A$\overrightarrow{C_n \odot K_1}$): A crown graph $C_n \odot K_1$, with n even in which the edges of the cycle are directed clockwise or anticlockwise and the pendent edges are oriented alternatively is called an alternating crown and is denoted as A$\overrightarrow{C_n \odot K_1}$.

2.23 Alternating incrown (Ai$\overrightarrow{C_n \odot K_1}$): A crown graph $C_n \odot K_1$, with n even in which the edges of the cycle are oriented alternatively and the pendent edges are directed towards the cycle is called an alternating incrown and is denoted as Ai$\overrightarrow{C_n \odot K_1}$.

2.24 Alternating outcrown (Ao$\overrightarrow{C_n \odot K_1}$): A crown graph $C_n \odot K_1$, with n even in which the edges of the cycle are oriented alternatively and the pendent edges are directed away from the cycle is called an alternating outcrown and is denoted as Ao$\overrightarrow{C_n \odot K_1}$.

2.25 Double alternating crown (DA$\overrightarrow{C_n \odot K_1}$): A crown graph $C_n \odot K_1$, with n even in which both the edges of the cycle and the pendent edges are oriented alternatively is called a double alternating crown and is denoted as DA$\overrightarrow{C_n \odot K_1}$.

2.26 Indragon (i$\overrightarrow{D_n m}$): A dragon graph $D_n m$ in which the edges of the cycle are directed clockwise or anticlockwise and the path edges are directed towards the cycle is called an indragon and is denoted as i$\overrightarrow{D_n m}$.

2.27 Outdragon (o$\overrightarrow{D_n m}$): A dragon graph $D_n m$ in which the edges of the cycle are directed clockwise or anticlockwise and the path edges are directed away from the cycle is called an outdragon and is denoted as o$\overrightarrow{D_n m}$.

2.28 Alternating dragon (A$\overrightarrow{D_n m}$): A dragon graph $D_n m$ in which the edges of the cycle are directed clockwise or anticlockwise and the path edges are oriented alternatively is called an alternating dragon and is denoted as A$\overrightarrow{D_n m}$.

2.29 Alternating indragon (Ai$\overrightarrow{D_n m}$): A dragon graph $D_n m$, with n even in which the edges of the cycle are oriented alternatively and the path edges are directed towards the cycle is called an alternating indragon and is denoted as Ai$\overrightarrow{D_n m}$.

2.30 Alternating outdragon (Ao$\overrightarrow{D_n m}$): A dragon graph $D_n m$, with n even in which the edges of the cycle are oriented alternatively and the path edges are directed away from the cycle is called an alternating outdragon and is denoted as Ao$\overrightarrow{D_n m}$.

2.31 Double alternating dragon (DA$\overrightarrow{D_n m}$): A dragon graph $D_n m$, with n even in which both the edges of the cycle and the path edges are oriented alternatively is called a double alternating dragon and is denoted as DA$\overrightarrow{D_n m}$.

3. MAIN RESULTS

3.1 Definition: Let D(p,q) be a digraph. A prime pair labeling of a digraph D is a labeling f:V→{1,2,…,p+q} which is both an in and outdegree prime pair labeling.

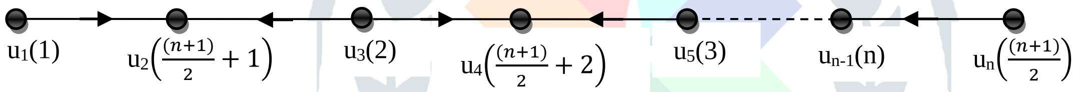
3.2 Observation: (1): If G is a graph such that N^+(u) & N^-(u) are either Φ or a singleton set then, G admits prime pair labeling.**(2):** By (1), Directed path ($\overrightarrow{P_n}$) and Directed cycle ($\overrightarrow{C_n}$) admits prime pair labeling.

3.3 Theorem: Alternating path (A$\overrightarrow{P_n}$) admits prime pair labeling.

Proof: Let V(A$\overrightarrow{P_n}$) = $\{u_i \mid 1 \le i \le n\}$ be the vertex set.

Case(i) n is odd

Here, A(A$\overrightarrow{P_n}$) = $\{\overleftrightarrow{u_{2i-1}u_{2i}} \mid 1 \le i \le \frac{n-1}{2}\} \cup \{\overleftrightarrow{u_{2i}u_{2i+1}} \mid 1 \le i \le \frac{n-1}{2}\}$ is the arc set



Define f:V→{1,2,3,…,2n-1} by $f(u_{2i-1}) = i$ for $1 \le i \le \frac{n+1}{2}$ and $f(u_{2i}) = \frac{n+1}{2}$ + i for $1 \le i \le \frac{n-1}{2}$

$N^-(u_{2i-1})$ = Φ for $1 \le i \le \frac{n+1}{2}$ ------(1)

$N^-(u_{2i}) = \{u_{2i-1}, u_{2i+1}\}$ for $1 \le i \le \frac{n-1}{2}$ and so gcd[$f(u_{2i-1}), f(u_{2i+1})$] = gcd[i,i+1] = 1 for $1 \le i \le \frac{n-1}{2}$

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^-(u_{2i})$ for $1 \le i \le \frac{n-1}{2}$ ---------(2)

From (1) & (2) f is an indegree prime pair labeling.

Further, $N^+(u_{2i-1}) = \{u_{2i-2}, u_{2i}\}$ for $2 \le i \le \frac{n-1}{2}$; $N^+(u_{2i})$ = Φ for $1 \le i \le \frac{n-1}{2}$ --------(3)

$N^+(u_1) = \{u_2\}$ & $N^+(u_n) = \{u_{n-1}\}$ ------(4)

Further, gcd[$f(u_{2i-2}), f(u_{2i})$] = gcd[$\frac{(n+1)}{2}$ + i −1, $\frac{(n+1)}{2}$ + i] = 1 for $2 \le i \le \frac{n-1}{2}$

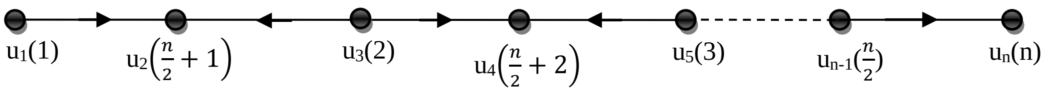
∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^+(u_{2i-1})$ for $2 \le i \le \frac{n-1}{2}$ ------(5)

From (3), (4) & (5) f is an outdegree prime pair labeling.

∴ f is a prime pair labeling.

Case(ii) n is even

Here, A(A$\overrightarrow{P_n}$) = $\{\overleftrightarrow{u_{2i-1}u_{2i}} \mid 1 \le i \le \frac{n}{2}\} \cup \{\overleftrightarrow{u_{2i}u_{2i+1}} \mid 1 \le i \le \frac{n-2}{2}\}$ is the arc set.



Define f:V→{1,2,3,…,2n-1} by $f(u_{2i-1}) = i$ for $1 \le i \le \frac{n}{2}$ and $f(u_{2i}) = \frac{n}{2}$ + i for $1 \le i \le \frac{n}{2}$

$N^-(u_{2i-1})$ = Φ for $1 \le i \le \frac{n}{2}$ -----------(7)

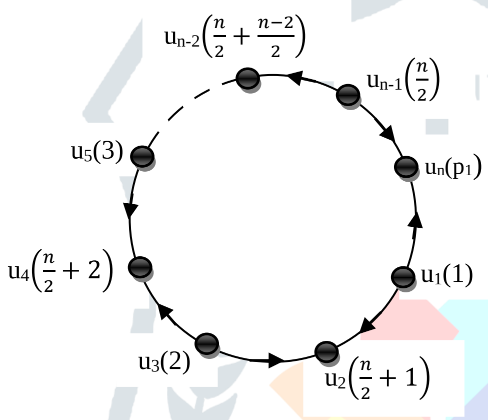
$N^-(u_{2i}) = \{u_{2i-1}, u_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and $N^-(u_n) = \{u_{n-1}\}$ ---------(8)

Further, gcd[$f(u_{2i-1}), f(u_{2i+1})$] = gcd[i,i+1] = 1 for $1 \le i \le \frac{n-2}{2}$

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{-}(u_{2i})$ for 1 ≤ i ≤ $\frac{n-2}{2}$ − − − − − − − − −(9)
From (7), (8) & (9) f is an indegree prime pair labeling.
$N^{+}(u_1) = \{u_2\}$ & $N^{+}(u_{2i})$ = Φ for 1 ≤ i ≤ $\frac{n}{2}$ − − − − − − − −(10)
$N^{+}(u_{2i-1}) = \{u_{2i-2}, u_{2i}\}$ for 2 ≤ i ≤ $\frac{n}{2}$ and so gcd[$f(u_{2i-2}), f(u_{2i})$] = gcd[$\frac{n}{2}$ + i −1, $\frac{n}{2}$ + i] = 1 for 2 ≤ i ≤ $\frac{n}{2}$
∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{+}(u_{2i-1})$ for 2 ≤ i ≤ $\frac{n}{2}$ − − − − − − − − −(11)
From (10) & (11) f is an outdegree prime pair labeling
∴ f is a prime pair labeling.
∴ From case(i) & case(ii), f is a prime pair labeling of A$\overrightarrow{P_n}$
A$\overrightarrow{P_n}$ admits prime pair labeling.

3.4 Theorem: Alternating cycle (A$\overrightarrow{C_n}$) admits prime pair labeling.

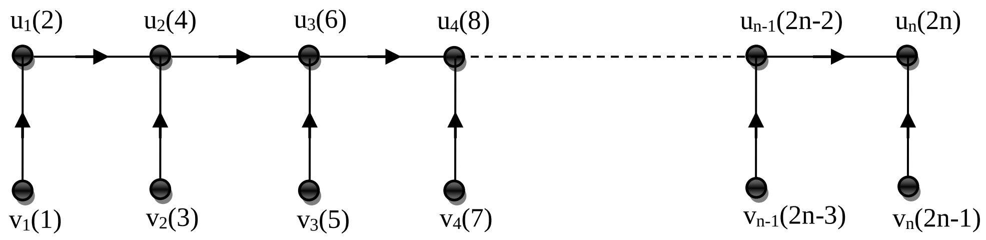
Proof: Let V(A$\overrightarrow{C_n}$) = $\{u_i \mid 1 \le i \le n\}$ be the vertex set and
A(A$\overrightarrow{C_n}$) = $\{\overrightarrow{u_{2i-1}u_{2i}} \mid 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{u_{2i}u_{2i+1}} \mid 1 \le i \le \frac{n-2}{2}\} \cup \{\overrightarrow{u_1u_n}\}$ be the arc set.
Let p_1 be the first prime number between n and 2n. Such a prime exists by 1.1.



Define f:V→{1,2,3,…,2n} by $f(u_{2i-1})$ = i for 1 ≤ i ≤ $\frac{n}{2}$,
$f(u_{2i}) = \frac{n}{2}$ + i for 1 ≤ i ≤ $\frac{n-2}{2}$ and $f(u_n) = p_1$
$N^{-}(u_{2i-1})$ = Φ for 1 ≤ i ≤ $\frac{n}{2}$ − − − − − − − −(1)
$N^{-}(u_{2i}) = \{u_{2i-1}, u_{2i+1}\}$ for 1 ≤ i ≤ $\frac{n-2}{2}$
$N^{-}(u_n) = \{u_1, u_{n-1}\}$
Further, gcd[$f(u_{2i-1}), f(u_{2i+1})$] = gcd[i,i+1] = 1 for 1 ≤ i ≤ $\frac{n-2}{2}$
∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{-}(u_{2i})$ for 1 ≤ i ≤ $\frac{n-2}{2}$ − − − − − − −(2)
Further gcd[$f(u_1), f(u_{n-1})$] = gcd[1,$\frac{n}{2}$] = 1 and so gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{-}(u_n)$ − − − − − − − −(3)
From (1), (2) & (3) f is an indegree prime pair labeling.
$N^{+}(u_1)=\{u_2, u_n\}$. Also, gcd[$f(u_2), f(u_n)$]=gcd[$\frac{n}{2}$ + 1, p_1] = 1 and so gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{+}(u_1)$ − − −(4)
$N^{+}(u_{2i})$ = Φ for 1 ≤ i ≤ $\frac{n}{2}$ − − − − − − − −(5)
$N^{+}(u_{2i-1}) = \{u_{2i-2}, u_{2i}\}$ for 2 ≤ i ≤ $\frac{n}{2}$ and also gcd[$f(u_{2i-2}), f(u_{2i})$] = gcd[$\frac{n}{2}$ + i −1, $\frac{n}{2}$ + i] = 1 for 2 ≤ i ≤ $\frac{n}{2}$
∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^{+}(u_{2i-1})$ for 2 ≤ i ≤ $\frac{n}{2}$ − − − − − − − − −(6)
From (4), (5) & (6) f is an outdegree prime pair labeling
∴ f is a prime pair labeling of A$\overrightarrow{C_n}$
Hence A$\overrightarrow{C_n}$ admits prime pair labeling.

3.5 Theorem: Upcomb (Up$\overrightarrow{P_n \odot K_1}$) admits prime pair labeling.

Proof: Let V(Up$\overrightarrow{P_n \odot K_1}$) = $\{u_i, v_i \mid 1 \le i \le n\}$ be the vertex set where u_i's and v_i's represents the vertices of the path and the i^{th} copy of K_1.
Then, A(Up$\overrightarrow{P_n \odot K_1}$) = $\{\overrightarrow{u_iu_{i+1}} \mid 1 \le i \le n-1\} \cup \{\overrightarrow{v_iu_i} \mid 1 \le i \le n\}$ is the arc set.



The digraph has 2n vertices and 2n-1 arcs.

Define f:V→{1,2,3,…,4n-1} by $f(u_i) = 2i$ for 1 ≤ i ≤ n and $f(v_i) = 2i - 1$ for 1 ≤ i ≤ n

$N^-(u_1) = \{v_1\}$ − − − − − − − −(1)

$N^-(u_i) = \{u_{i-1}, v_i\}$ for 2 ≤ i ≤ n. Also, gcd[$f(u_{i-1}), f(v_i)$] = gcd[2i−2, 2i−1] = 1 for 2 ≤ i ≤ n

∴ gcd[f(v),f(w)] = 1 ∀ v,w ∈ $N^-(u_i)$ for 2 ≤ i ≤ n − − − − − − − −(2)

$N^-(v_i)$ = Φ for 1 ≤ i ≤ n − − − − − − − −(3)

From (1), (2) & (3) f is an indegree prime pair labeling.

$N^+(u_i) = \{u_{i+1}\}$ for 1 ≤ i ≤ n−1 ; $N^+(u_n)$ = Φ ; $N^+(v_i) = \{u_i\}$ for 1 ≤ i ≤ n.

∴ N^+(w) contains exactly one element ∀ w ∈ D.

∴ f is an outdegree prime pair labeling.

∴ f is a prime pair labeling of Up$\overrightarrow{P_n \odot K_1}$.

Hence Up$\overrightarrow{P_n \odot K_1}$ admits prime pair labeling.

Conclusion: In this way, we tested the digraphs including all types of comb, crown, dragon and some more for the existence of prime pair labeling.

References:

[1] Arumugam S and Ramachandran S, "Invitation to Graph Theory", SCITECH Publications (India) PVT.LTD.

[2] https://en.m.wikipedia.org/wiki/Bertrand%27s_postulate.

[3] http://mathworld.wolfram.com.

[4] Kourosh Eshgi "Introduction to Graceful Graphs", September 2002, © Copyright by Sharif University of Technology.

[5] Palani K, Selvalakshmi M and Catherine A, "Indegree Prime Pair Labeling of Directed Graphs", Enrich, IX(II): 22-28, Jan-Jun 2018.

[6] Palani K, Catherine A and Suganya G, "Outdegree Prime Pair Labeling of Directed Graphs", Journal of Emerging Technologies and Innovative Research(JETIR), Vol 6, Issue 1, January 2019, pp.442-448.

[7] Tout A, .Dabboucy A N, Howalla K, "Prime Labeling of Graphs", Nat.Acad.Sci.Letters, 11(1982), 365-368.