Numerical Solution of Fuzzy neutral delay differential equations

An application Runge-Kutta method of order four

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Abstract : We are solving Fuzzy Neutral Delay Differential Equations by an application of the fourth order Runge-Kutta method. With a numerical example we verify the volatility of the theory and graphical illustration is provided to compare the approximate and analytical solutions of the solved problem**.**

IndexTerms- Fuzzy, Neutral Delay, Differential Equations, Runge-Kutta Method, Numerical Solutions

I. INTRODUCTION

Fuzzy set theory introduced by Lotfi Zadeh, in 1960 is used as a powerful tool for modeling uncertainty in mathematical models. In Recent times, Fuzzy differential equations are attaining more importance in the field of pure and applied mathematics. The fuzzy differential equations (FDEs) and the initial value problem where regularly treated by Kaleva [8, 9] and by Seikkala [15]. The numerical method for solving fuzzy differential equations is introduced by Ma, Friedmen, and Kandedl [10] by the standard Euler method and by authors Abbasbandy et.al.,[1], by Taylor method. In the last few years many works have been performed by several authors in numerical solutions of fuzzy differential equations [1, 2, 4, 5 and 6]. Abbasbandy and Allahviranloo [2] ,Pederson and Sambandham [11] have investigated the numerical method for solving fuzzy differential equation by Runge-Kutta method of order four and numerical solution of hybrid fuzzy differential equation by using Runge-Kutta method respectively. Al-Rawi et,al., [16] have provided a numerical method for solving Delay differential equations by Runge-Kutta method of order four. Alfredo Bellan and Marino Zennaro [3] studied Numerical methods for delay differential equations in detail. In this article, we addressing comfortable fuzzy RK-4 method for fuzzy neutral delay differential equation by small changes in the Runge-Kutta method of order four [16]. In Section **II**, we review fuzzy neutral delay differential systems. In Section **III,** the Runge-Kutta method of order four for approaching fuzzy neutral delay differential equations is discussed. Section **IV** contains a numerical example to illustrate the theory.

II. FUZZY NEUTRAL DELAY DIFFERENTIAL EQAUTIONS

Neutral delay differential equations (NDDEs) are considered as a branch of delay differential equations (DDEs). DDEs arise in many areas of various mathematical modeling. For instance; infectious diseases, population dynamics, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircrafts and control problems are modeled by fuzzy differential equations with delay. The Neutral delay differential equations are transformed into delay differential equation because delay differential equations are similar to ordinary differential equations, but their evolution involves past values of the time and state variable. The solution of delay differential equations requires knowledge of not only the current state, but also of the state at a certain time previously. Neutral delay differential equations (NDDEs) have numerous applications in science and engineering as discussed by Jayakumar et.al, [7]. Prasantha Bharathi et.al.,[12,13 and 14] elaborately studied multiple delays. Since we are studying neutral delay differential equations, the initial function is defined over the interval $[-\tau ;0]$ and it is mapped into a solution curve on the interval $[0;\tau]$.
Consider the FNDDE,

$$\begin{cases} y'(t) = f(t, y(t), y'(t-\tau)), & t_0 \le t \le t_f \\ y(t) = \phi(t), & t \le t_0 \end{cases} \tag{1}$$

Where, $f : [t_0, t_f] \times E^1 \times E^1 \to E^1$

The above equation (1) is transformed in to following fuzzy valued function

$$[f(\text{t}, \text{y(t)}, \text{y}'(\text{t}-\tau),]^{\alpha} = \begin{matrix} [\min\{ f(t,u(t),\text{u}'(\text{t}-\tau)) : u(t) \in [\underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t)], u'(\text{t}-\tau) \in [\underline{y}^{\alpha}(\text{t}-\tau), \overline{y}^{\alpha}(\text{t}-\tau)]\}, \\ \max\{ f(t,u(t),\text{u}'(\text{t}-\tau)) : u(t) \in [\underline{y}^{\alpha}(t), \overline{y}^{\alpha}(t)], u'(\text{t}-\tau) \in [\underline{y}^{\alpha}(\text{t}-\tau), \overline{y}^{\alpha}(\text{t}-\tau)]\}] \end{matrix} \tag{2}$$

.

The equation (1) is transformed in to fuzzy delay differential equation

$$\begin{cases} y'(t) = f(t,\phi(t),\phi(t-\tau)), & t_0 \le t \le t_n \\ \phi(t) = \xi(t) & -\tau \le t \le t_0 \\ \phi(t_0) = \xi(t_0) = \xi_0 \end{cases} \quad (3)$$

The corresponding equation of (3) is given by

$$[f(t,\phi(t),\phi(t-\tau))]^\alpha = \begin{matrix} [\min\{f(t,v(t),v(t-\tau)) : v(t) \in [\underline{\phi}^\alpha(t), \overline{\phi}^\alpha(t)], v(t-\tau) \in [\underline{\phi}^\alpha(t-\tau_1), \overline{\phi}^\alpha(t-\tau_1)],)],\}, \\ \max\{f(t,u(t),u(t-\tau)) : v(t) \in [\underline{\phi}^\alpha(t), \overline{\phi}^\alpha(t)], v(t-\tau_1) \in [\underline{\phi}^\alpha(t-\tau_1), \overline{\phi}^\alpha(t-\tau_1)]\}] \end{matrix} \quad (4)$$

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III. FOURTH ORDER RUNGE-KUTTA METHOD

We shall re-define the Runge-Kutta method for the above fuzzy valued function, i.e., for fuzzy delay differential equations as follows

$$\underline{K}_1(t;\phi(t;\alpha)) = \min\{hf(t,\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{\phi}(t_{k,n};\alpha), \overline{\phi}(t_{k,n};\alpha)], \phi(t-\tau) \in [\underline{\phi}(t_{k,n}-\tau;\alpha), \overline{\phi}(t_{k,n}-\tau;\alpha)]\}$$

$$\overline{K}1(t;\phi(t;\alpha)) = \max\{hf(t,\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{\phi}(t_{k,n};\alpha), \overline{\phi}(t_{k,n};\alpha)], \phi(t-\tau) \in [\underline{\phi}(t_{k,n}-\tau;\alpha), \overline{\phi}(t_{k,n}-\tau;\alpha)]\}$$

$$\underline{K}_2(t;\phi(t;\alpha)) = \min\{hf(t+\frac{h}{2},\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{z}_1(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}1(t_{k,n},\overline{\phi}(t_{k,n};\alpha))],$$
$$\phi(t-\tau) \in [\underline{z}_1(t_{k,n}-\tau,\underline{\phi}(t_{k,n}-\tau;\alpha)), \overline{z}1(t_{k,n}-\tau,\overline{\phi}(t_{k,n}-\tau;\alpha))]]\}$$

$$\overline{K}2(t;\phi(t;\alpha)) = \max\{hf(t+\frac{h}{2},\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{z}_1(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}1(t_{k,n},\overline{\phi}(t_{k,n};\alpha))],$$
$$\phi(t-\tau) \in [\underline{z}_1(t_{k,n}-\tau_1,\underline{\phi}(t_{k,n}-\tau;\alpha)), \overline{z}1(t_{k,n}-\tau_1,\overline{\phi}(t_{k,n}-\tau;\alpha))]]\}$$

$$\underline{K}_3(t;\phi(t;\alpha)) = \min\{hf(t+\frac{h}{2},\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{z}_1(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}1(t_{k,n},\overline{\phi}(t_{k,n};\alpha))],$$
$$\phi(t-\tau) \in [\underline{z}_2(t_{k,n}-\tau_1,\underline{\phi}(t_{k,n}-\tau_1;\alpha)), \overline{z}2(t_{k,n}-\tau,\overline{\phi}(t_{k,n}-\tau;\alpha))]]\}$$

$$\overline{K}3(t;\phi(t;\alpha)) = \max\{hf(t+\frac{h}{2},\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{z}_1(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}1(t_{k,n},\overline{\phi}(t_{k,n};\alpha))],$$
$$\phi(t-\tau) \in [\underline{z}_2(t_{k,n}-\tau,\underline{\phi}(t_{k,n}-\tau;\alpha)), \overline{z}2(t_{k,n}-\tau,\overline{\phi}(t_{k,n}-\tau;\alpha))]]\}$$

$$\underline{K}_4(t;\phi(t;\alpha)) = \min\{hf(t+h,\phi(t),\phi(t-\tau)) \mid \phi(t) \in [\underline{z}_3(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}3(t_{k,n},\overline{\phi}(t_{k,n};\alpha))],$$
$$\phi(t-\tau) \in [\underline{z}_3(t_{k,n}-\tau_1,\underline{\phi}(t_{k,n}-\tau_1;\alpha)), \overline{z}3(t_{k,n}-\tau_1,\overline{\phi}(t_{k,n}-\tau_1;\alpha))]\}$$

$$\overline{K}4(t;\phi(t;\alpha)) = \max\{hf(t+h,\phi(t),\phi(t-\tau_1),\phi(t) \in [\underline{z}_3(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}3(t_{k,n},\overline{\phi}(t_{k,n};\alpha))],$$
$$\phi(t-\tau) \in [\underline{z}_3(t_{k,n}-\tau,\underline{\phi}(t_{k,n}-\tau_1;\alpha)), \overline{z}3(t_{k,n}-\tau,\overline{\phi}(t_{k,n}-\tau;\alpha))]]\}$$

Where,

$$\underline{z}_1(t_{k,n},\underline{\phi}(t_{k,n};\alpha)) = \underline{\phi}(t_{k,n};\alpha) + \frac{1}{2}\underline{K}_1(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}1(t_{k,n},\overline{\phi}(t_{k,n};\alpha)) = \overline{\phi}(t_{k,n};\alpha) + \frac{1}{2}\overline{K}1(t_{k,n},\overline{\phi}(t_{k,n};\alpha)),$$

$$\underline{z}_2(t_{k,n},\underline{\phi}(t_{k,n};\alpha)) = \underline{\phi}(t_{k,n};\alpha) + \frac{1}{2}\underline{K}_2(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}2(t_{k,n},\overline{\phi}(t_{k,n};\alpha)) = \overline{\phi}(t_{k,n};\alpha) + \frac{1}{2}\overline{K}2(t_{k,n},\overline{\phi}(t_{k,n};\alpha)),$$

$$\underline{z}_3(t_{k,n},\underline{\phi}(t_{k,n};\alpha)) = \underline{\phi}(t_{k,n};\alpha) + \underline{K}_3(t_{k,n},\underline{\phi}(t_{k,n};\alpha)), \overline{z}3(t_{k,n},\overline{\phi}(t_{k,n};\alpha)) = \overline{\phi}(t_{k,n};\alpha) + \overline{K}3(t_{k,n},\overline{\phi}(t_{k,n};\alpha)),$$

Then the approximate solution is given by

$$\underline{y}(t_{n+1};\alpha) = \underline{\phi}(t_{n+1};\alpha) + \underline{y}(t_n-\tau;\alpha), \overline{y}(t_{n+1};\alpha) = \overline{\phi}(t_{n+1};\alpha) + \overline{y}(t_n-\tau;\alpha).$$

Where, $$\begin{cases} \underline{\phi}(t_{n+1};\alpha) = \underline{\phi}(t_n;\alpha) + \underline{\phi}(t_n - \tau;\alpha) + \frac{1}{6}\underline{K}_1(t;\phi(t;\alpha)) + 2\underline{K}_2(t;\phi(t;\alpha)) + 2\underline{K}_3(t;\phi(t;\alpha)) + \underline{K}_4(t;\phi(t;\alpha)), \\ \overline{\phi}(t_{n+1};\alpha) = \overline{\phi}(t_n;\alpha) + \overline{\phi}(t_n - \tau;\alpha) + \frac{1}{6}\overline{K}1(t;\phi(t;\alpha)) + 2\overline{K}2(t;\phi(t;\alpha)) + 2\overline{K}3(t;\phi(t;\alpha)) + \overline{K}4(t;\phi(t;\alpha)). \end{cases}$$

IV. NUMERICAL EXAMPLE

In this section, we will discuss a example to support the theory,

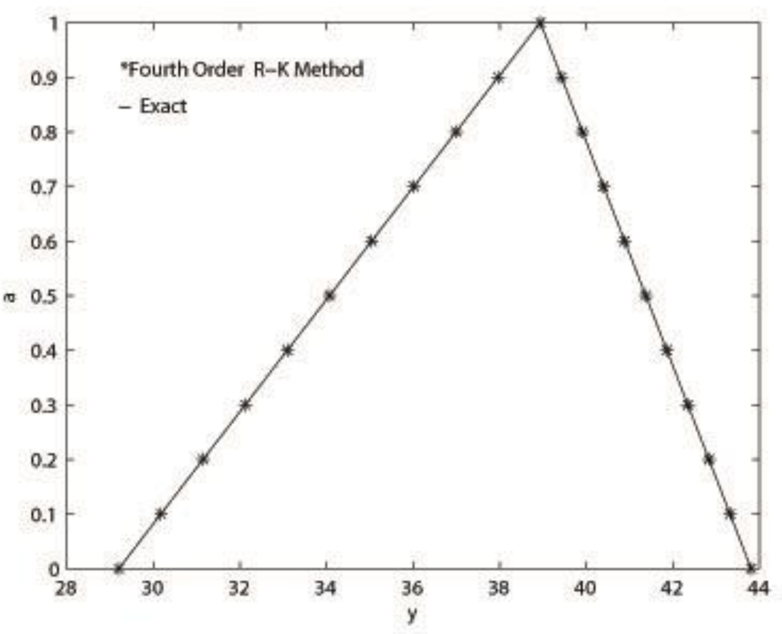
$$\begin{cases} y'(t;\alpha) = (0.75 + 0.25\alpha)y(t) + y'(t-1), (1.125 - 0.125\alpha)y(t) + y'(t-1), & 0 \le t \le 3 \\ y(t;\alpha) = (0.75 + 0.25\alpha), (1.125 - 0.125\alpha), & -1 \le t \le 0 \end{cases}$$

The exact solution is given by

$$Y(t) = \begin{cases} [(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t] & (0 \le t \le 1) \\ [(0.75 + 0.25\alpha)(e^t + (t-1)e^{t-1}), (1.125 - 0.125\alpha)(e^t + (t-1)e^{t-1})], & (1 \le t \le 2) \\ [(0.75 + 0.25\alpha)(e^t + e^{t-1} + \frac{1}{2}(t-2)(t+2e)e^{t-2}), (1.125 - 0.125\alpha)(e^t + e^{t-1} + \frac{1}{2}(t-2)(t+2e)e^{t-2})], & (2 \le t \le 3) \end{cases}$$

The approximate solutions are found by RK-4 method as shown in (**III**). Both approximate and exact solutions are tabulated and plotted below.

Approximate & Analytic solutions of FNDDE.				
t=3,	Approximate Solution		Analytic Solution	
α	$\underline{y}(t;\alpha)$	$\overline{y}(t;\alpha)$	$\underline{Y}(t;\alpha)$	$\overline{Y}(t;\alpha)$
0	29.2056777554927	43.8085166332391	29.2058038978031	43.8087058467047
0.1	30.1792003473425	43.3217553373142	30.1793306943966	43.3219424484080
0.2	31.1527229391922	42.8349940413893	31.1528574909900	42.8351790501113
0.3	32.1262455310420	42.3482327454645	32.1263842875835	42.3484156518146
0.4	33.0997681228918	41.8614714495396	33.0999110841769	41.8616522535178
0.5	34.0732907147415	41.3747101536147	34.0734378807703	41.3748888552211
0.6	35.0468133065913	40.8879488576898	35.0469646773638	40.8881254569244
0.7	36.0203358984410	40.4011875617649	36.0204914739572	40.4013620586277
0.8	36.9938584902908	39.9144262658401	36.9940182705507	39.9145986603310
0.9	37.9673810821406	39.4276649699152	37.9675450671441	39.4278352620342
1.0	38.9409036739903	38.9409036739903	38.9410718637375	38.9410718637375



V. CONCLUSION

In this paper, we have applied iterative solution by fourth order Runge-Kutta method for finding the numerical solution of fuzzy neutral delay differential equations. In the proposed method the convergence order is $O(h^4)$ (found from table). The graphical representation also confirms that the RK method of order four is possibly provides better accuracy of solutions as compared with the exact solutions.

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