DIFFERENTIAL OF KING GRAPHS AND COMPLETE N-ARY TREES

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Abstract: Let $G=(V,E)$ be an arbitrary graph. For any subset X of V, let $B(X)$ be the set of all vertices in $V-X$, having neighbour in X. J.L. Mashburn et al. defined the *differential of a set* X to be $\partial(X)=|B(X)|-|X|$ and the *differential of a graph* to be equal to $\max\partial(X)$, for any subset of X of V. In this paper, we obtain differential value of classes of king graphs and complete n-ary trees.

I. INTRODUCTION

Let $G=(V,E)$ be a graph. For graph theoretical terminology not given here, refer to Harary $[2]$. For a vertex $v\in V$, the *open neighborhood* of v is the set $N(v)=\{u\in V | uv\in E\}$ and the *closed neighborhood* of the set $N[v]=N(v)\cup\{v\}$. For a set $X\subset V$, its open neighborhood is $N(X)=\underset{v\in X}{\cup}N(v)$ and the closed neighborhood is $N[X]=N(X)\cup X$.

The *boundary* $B(X)$ of a set X is defined to be the set of vertices in $V-X$ dominated by vertices in X, that is $B(X)=(V-X)\cap N(X)$. The *differential* $\partial(X)$ of X is $|B(X)|-|X|$. The *differential of a graph G* is defined as $\partial(G)=\max\{\partial(X)|X\subset V\}$. Let $T\subset V$ such that $\partial(G)=\partial(T)$ Then we say T as ∂-set. As reported in [4], the *differential of a set* was first defined by Hedetniemi [3], and later studied by Mashburn et al. [4] and Goddard and Henning [1]. The *minimum differential* of an *independent set* was also studied by Zhang [6].

In this paper, we obtain the differential value of classes of king graphs and complete n-ary tree.

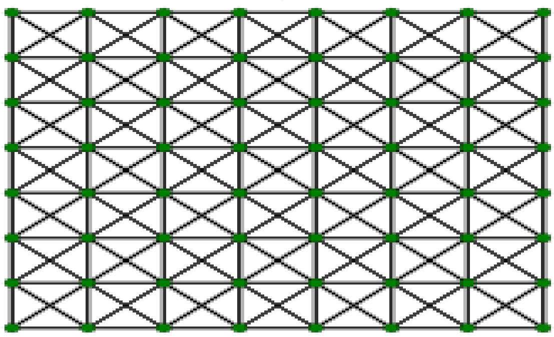

Figure 1 $K_{8\times8}$ - King graph

II. DIFFERENTIAL VALUE OF KING GRAPHS

DEFINITION 2.1: The $m\times n$ king graph is a graph with mn vertices in which each vertex represents a square in a $m\times n$ chessboard and each edge corresponds to legal move by a king. We denote $m\times n$ king graph as $K_{m\times n}$. A 8×8 king graph is given in figure 1.

THEOREM 2.2: For any $K_{m\times n}$ with $m\equiv 0\pmod 3$,
$$\partial(K_{m\times n})=\begin{cases}\dfrac{7mn}{9} & \text{when } n\equiv 0 \pmod 3\\ \dfrac{(7n-4)m}{9} & \text{when } n\equiv 1 \pmod 3\\ \dfrac{(7n-2)m}{9} & \text{when } n\equiv 2 \pmod 3\end{cases}$$

Proof. Let $V(K_{m\times n})=\{v_{11},v_{12},....,v_{1n},v_{21},v_{22},....v_{2n},...,v_{m1},v_{m2},...,v_{mn}\}$ and S be any ∂-set of $K_{m\times n}$. For any $K_{3\times 3}$ there is only one possible ∂-set. Clearly $S=\{v_{22}\}$ and $\partial(S)=8-1=7=\partial(K_{3\times 3})$.

Case (i) $n\equiv 0(\mathrm{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m}{3}\times\frac{n}{3}$ copies of $K_{3\times 3}$ and hence $\partial(K_{m\times n})=\left(\frac{7m}{3}\right)\left(\frac{n}{3}\right)=\frac{7mn}{9}$

Case (ii) $n\equiv 1(\mathrm{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m}{3}\times\frac{n-1}{3}$ copies of $K_{3\times 3}$ and a path P_m and clearly $\partial(P_m)=\left(\frac{m}{3}\right)$.

Hence $\partial(K_{m\times n})=7\left(\frac{m}{3}\right)\left(\frac{n-1}{3}\right)+\left(\frac{m}{3}\right)=\frac{(7n-4)m}{9}$

Case (iii) $n\equiv 2(\mathrm{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m}{3}\times\frac{n-2}{3}$ copies of $K_{3\times 3}$ and a $K_{m\times 2}$. Let $V(K_{m\times 2})=\{v_{1(n-1)},v_{1n},v_{2(n-1),}v_{2n},....,v_{m(n-1),}v_{mn}\}$ and clearly $\{v_{2n},v_{5n},....,v_{(m-1)n}\}$ is a ∂ - set of $K_{m\times 2}$. Hence $\partial(K_{m\times 2})=4\left(\frac{m}{3}\right)$. Therefore, $\partial(K_{m\times n})=7\left(\frac{m}{3}\right)\left(\frac{n-2}{3}\right)+4\left(\frac{m}{3}\right)=\frac{(7n-2)m}{9}$.

THEOREM 2.3: For any $K_{m\times n}$ with $m\equiv 1(\mathrm{mod}\,3)$,
$$\partial(K_{m\times n})=\begin{cases}\dfrac{7mn-4n}{9} & \text{when } n\equiv 0 \pmod 3\\ \dfrac{7mn-4m-4n+1}{9} & \text{when } n\equiv 1 \pmod 3\\ \dfrac{7mn-2m-4n-4}{9} & \text{when } n\equiv 2 \pmod 3\end{cases}$$

Proof. Let $V(K_{m\times n})=\{v_{11},v_{12},....,v_{1n},v_{21},v_{22},....v_{2n},...,v_{m1},v_{m2},...,v_{mn}\}$. Clearly $\partial(K_{3\times 3})=7$

Case (i) $n\equiv 0(\mathrm{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m-1}{3}\times\frac{n}{3}$ copies of $K_{3\times 3}$ and a path P_n. Let $V(P_n)=\{v_{m1},v_{m2},v_{m3},....v_{mn}\}$ and clearly $\partial(P_n)=\left(\frac{n}{3}\right)$. Therefore, $\partial(K_{m\times n})=7\left(\frac{m-1}{3}\right)\left(\frac{n}{3}\right)+\left(\frac{n}{3}\right)=\frac{(7m-4)n}{9}$

Case (ii) $n\equiv 1(\mathrm{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m-1}{3}\times\frac{n-1}{3}$ copies of $K_{3\times 3}$ and a path P_{m+n-1}. Let

$V\left(P_{m+n-1}\right)=\left\{v_{m1},v_{m2},v_{m3},....v_{m(n-1)},v_{1n},v_{2n},....,v_{(m-1)n},v_{mn}\right\}$ and $\partial\left(P_{m+n-1}\right)=\frac{m+n-2}{3}$. Therefore,

$\partial\left(K_{m\times n}\right)=7\left(\frac{m-1}{3}\right)\left(\frac{n-1}{3}\right)+\frac{m+n-2}{3}=\frac{7mn-4m-4n+1}{9}$.

Case (iii) $n\equiv 2(\text{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m-1}{3}\times\frac{n-2}{3}$ copies of $K_{3\times 3}$, a path P_n and a $K_{m\times 2}$. Let $V\left(P_n\right)=\left\{v_{m1},v_{m2},v_{m3},....v_{mn}\right\}$ and $\partial\left(P_n\right)=\left(\frac{n-2}{3}\right)$. Let $V\left(K_{m\times 2}\right)=\left\{v_{1(n-1)},.....,v_{m(n-1)},v_{1n},v_{2n},....,v_{mn}\right\}$ and clearly $\left\{v_{2n},v_{5n},....,v_{(m-2)n}\right\}$ is a ∂ - set of $K_{m\times 2}$. Hence $\partial\left(K_{m\times 2}\right)=4\left(\frac{m-1}{3}\right)$. Therefore,

$\partial\left(K_{m\times n}\right)=7\left(\frac{m-1}{3}\right)\left(\frac{n-2}{3}\right)+\left(\frac{n-2}{3}\right)+4\left(\frac{m-1}{3}\right)=\frac{7mn-2m-4n-4}{9}$.

THEOREM 2.4: For any $K_{m\times n}$ with $m\equiv 2(\text{mod}\,3)$,

$$\partial\left(K_{m\times n}\right)=\begin{cases}\frac{7mn-2n}{9} & \text{when } n\equiv 0\ (\text{mod}\,3)\\ \frac{7mn-4m-2n-4}{9} & \text{when } n\equiv 1\ (\text{mod}\,3)\\ \frac{7mn-2m-2n-2}{9} & \text{when } n\equiv 2\ (\text{mod}\,3)\end{cases}$$

Proof. Let $V\left(K_{m\times n}\right)=\left\{v_{11},v_{12},....,v_{1n},v_{21},v_{22},....v_{2n},...,v_{m1},v_{m2},...,v_{mn}\right\}$. Clearly, $\partial\left(K_{3\times 3}\right)=7$

Case (i) $n\equiv 0(\text{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m-2}{3}\times\frac{n}{3}$ copies of $K_{3\times 3}$ and a $K_{2\times n}$.

Let $V\left(K_{2\times n}\right)=\left\{v_{(m-1)1},v_{(m-1)2},....,v_{(m-1)n},v_{m1},v_{m2},v_{m3},....v_{mn}\right\}$ and clearly $\left\{v_{m2},v_{m5},....,v_{m(n-1)}\right\}$ is a ∂ - set of $K_{2\times n}$.

Hence $\partial\left(K_{2\times n}\right)=4\left(\frac{n}{3}\right)$.Therefore, $\partial\left(K_{m\times n}\right)=\left(\frac{7m-2}{3}\right)\left(\frac{n}{3}\right)=\frac{7mn-2n}{9}$.

Case (ii) $n\equiv 1(\text{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m-2}{3}\times\frac{n-1}{3}$ copies of $K_{3\times 3}$ a $K_{2\times n}$ and a path P_{m-2}. Let

$V\left(K_{2\times n}\right)=\left\{v_{(m-1)1},v_{(m-1)2},....,v_{(m-1)n},v_{m1},v_{m2},v_{m3},....v_{mn}\right\}$. Clearly $\left\{v_{m2},v_{m5},....,v_{m(n-1)}\right\}$ is a ∂ - set of $K_{2\times n}$ and

$\partial\left(K_{2\times n}\right)=4\left(\frac{n-1}{3}\right)$. Let $V\left(P_{m-2}\right)=\left\{v_{1n},v_{2n},v_{3n},....v_{(m-2)n}\right\}$ and $\partial\left(P_{m-2}\right)=\left(\frac{m-2}{3}\right)$. Therefore,

$\partial\left(K_{m\times n}\right)=7\left(\frac{m-2}{3}\right)\left(\frac{n-1}{3}\right)+4\left(\frac{n-1}{3}\right)+\left(\frac{m-2}{3}\right)=\frac{7mn-4m-2n-4}{9}$

Case (iii) $n\equiv 2(\text{mod}\,3)$

$K_{m\times n}$ contains mutually disjoint $\frac{m-2}{3}\times\frac{n-2}{3}$ copies of $K_{3\times3}$, a $K_{2\times n}$ and a $K_{(m-2)\times n}$. Let $V(K_{2\times n})=\{v_{(m-1)1},v_{(m-1)2},....,v_{(m-1)n},v_{m1},v_{m2},v_{m3},....v_{mn}\}$. Clearly $\{v_{m2},v_{m5},....,v_{mn}\}$ is a ∂ - set of $K_{2\times n}$ and $\partial(K_{2\times n})=4\left(\frac{n-2}{3}\right)+2=\frac{4n-2}{3}$. Let $V(K_{(m-2)\times n})=\{v_{1(n-1)},v_{2(n-1)},....,v_{(m-2)(n-1)},v_{1n},v_{2n},...,v_{(m-2)n}\}$. Clearly $\{v_{2n},v_{5n},....,v_{(m-3)n}\}$ is a ∂ - set of $K_{(m-2)\times n}$ and hence $\partial(K_{(m-2)\times n})=4\left(\frac{m-2}{3}\right)$. Therefore,

$$\partial(K_{m\times n})=7\left(\frac{m-2}{3}\right)\left(\frac{n-2}{3}\right)+\left(\frac{4n-2}{3}\right)+4\left(\frac{m-2}{3}\right)=\frac{7mn-2m-2n-2}{9}.$$

III. DIFFERENTIAL OF COMPLETE N-ARY TREES

DEFINITION 3.1: A n *-ary tree* is a rooted tree in which each node has no more than n children. A binary tree is a special case where $n=2$. A *complete* n *-ary* tree is a n -ary tree in which each node has exactly n children.

In $[5]$, the differential value for complete binary tree was obtained. We extend the result to complete n -ary tree.

THEOREM 3.2: For any complete n -ary tree G with k levels where $k\geq1$

$$\partial(G)=\begin{cases}\frac{r^k-1}{r-1}\left[\frac{r^3-r^2+r}{r^2+r+1}\right] & \text{if } k\equiv0(\text{mod }3)\\ \frac{r^{k+1}-1}{r-1}-2\left[\frac{r^{k+2}-1}{r^3-1}\right] & \text{if } k\equiv1(\text{mod }3)\\ \frac{r^{k+1}-1}{r-1}\left[\frac{r^2-r+1}{r^2+r+1}\right] & \text{if } k\equiv2(\text{mod }3)\end{cases}$$

Proof: Let G be a complete n-ary tree. Let S_i be the set of all vertices in level i and $|S_i|=n^i$.

Clearly $S_{k-1}\cup S_{k-4}\cup....\cup S_2$ is a ∂ -set of G when $k\equiv0(\text{mod }3)$, $S_{k-1}\cup S_{k-4}\cup....\cup S_0$ is a ∂ -set of G when $k\equiv1(\text{mod }3)$ and $S_{k-1}\cup S_{k-4}\cup....\cup S_1$ is a ∂ -set of G when $k\equiv2(\text{mod }3)$

Case (i) $k\equiv0(\text{mod }3)$

$$\left(r^k+r^{k-2}+r^{k-3}+r^{k-5}+r^{k-6}+....+r^3+r\right)-\left(r^{k-1}+r^{k-4}+....+r^5+r^2\right)$$

$$=r^k\left(1+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^5}+\frac{1}{r^6}+....+\frac{1}{r^{k-3}}+\frac{1}{r^{k-1}}\right)-\left(r^{k-1}+r^{k-4}+....+r^5+r^2\right)$$

$$=r^k\left(1+\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\left(\frac{1}{r}\right)^4+....+\left(\frac{1}{r}\right)^{k-1}\right)-2\left(r^{k-1}+r^{k-4}+....+r^5+r^2\right)$$

$$=r^k\left(\frac{1-\left(\frac{1}{r}\right)^k}{1-\frac{1}{r}}\right)-2r^k\left(\frac{1}{r}+\frac{1}{r^4}+\frac{1}{r^7}+.....+\frac{1}{r^{k-5}}+\frac{1}{r^{k-2}}\right)$$

$$=r^k\left(\frac{r^k-1}{r-1}\times\frac{r}{r^k}\right)-2\frac{r^k}{r}\left(\frac{1}{r^0}+\frac{1}{r^3}+\frac{1}{r^6}+.....+\frac{1}{r^{k-6}}+\frac{1}{r^{k-3}}\right)$$

$$= r\left(\frac{r^k-1}{r-1}\right)-2\frac{r^k}{r}\left(1+\left(\frac{1}{r}\right)^3+\left(\frac{1}{r}\right)^6+.....+\left(\frac{1}{r}\right)^{k-3}\right)$$

$$= r\left(\frac{r^k-1}{r-1}\right)-2\frac{r^k}{r}\left(\frac{1-\left(\frac{1}{r^3}\right)^{\frac{k}{3}}}{1-\frac{1}{r^3}}\right)$$

$$= r\left(\frac{r^k-1}{r-1}\right)-2\frac{r^k}{r}\left(\frac{\left(r^3\right)^{\frac{k}{3}}-1}{r^3-1}\times\frac{r^3}{r^k}\right)$$

$$= r\left(\frac{r^k-1}{r-1}\right)-2r^2\left(\frac{r^k-1}{r^3-1}\right)$$

$$= r\left(\frac{r^k-1}{r-1}\right)-2r^2\left(\frac{r^k-1}{(r-1)\left(r^2+r+1\right)}\right)$$

$$= \frac{r^k-1}{r-1}\left(r-\frac{2r^2}{\left(r^2+r+1\right)}\right)$$

$$= \frac{r^k-1}{r-1}\left(\frac{r^3-r^2+r}{r^2+r+1}\right)$$

Case (ii) $k\equiv 1(\bmod 3)$

$$\left(r^k+r^{k-2}+r^{k-3}+r^{k-5}+r^{k-6}+....+r^2+r\right)-\left(r^{k-1}+r^{k-4}+r^{k-7}+....+r^3+r^0\right)$$

$$= r^k\left(1+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^5}+\frac{1}{r^6}+....+\frac{1}{r^{k-2}}+\frac{1}{r^{k-1}}\right) - \left(r^{k-1}+r^{k-4}+r^{k-7}+....+r^3+r^0\right)$$

$$= r^k\left(1+\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\left(\frac{1}{r}\right)^4+....+\left(\frac{1}{r}\right)^k\right) - 2\left(r^{k-1}+r^{k-4}+r^{k-7}+.......+r^3+r^0\right)$$

$$= r^k\left(\frac{1-\left(\frac{1}{r}\right)^{k+1}}{1-\frac{1}{r}}\right)-2r^k\left(\frac{1}{r}+\frac{1}{r^4}+\frac{1}{r^7}+.....+\frac{1}{r^{k-3}}+\frac{1}{r^k}\right)$$

$$= r^k\left(\frac{r^{k+1}-1}{r-1}\times\frac{r}{r^{k+1}}\right)-2r^k\left(\frac{1}{r}+\frac{1}{r^4}+\frac{1}{r^7}+.....+\frac{1}{r^{k-3}}+\frac{1}{r^k}\right)$$

$$= \left(\frac{r^{k+1}-1}{r-1}\right)-2\frac{r^k}{r}\left(1+\frac{1}{r^3}+\frac{1}{r^6}+.....+\frac{1}{r^{k-2}}+\frac{1}{r^{k-1}}\right)$$

$$= \left(\frac{r^{k+1}-1}{r-1}\right)-2\frac{r^k}{r}\left(\frac{1-\left(\frac{1}{r^3}\right)^{\frac{k+2}{3}}}{1-\frac{1}{r^3}}\right)$$

$$= \left(\frac{r^{k+1}-1}{r-1}\right) - 2\frac{r^k}{r}\left(\frac{r^{k+2}-1}{r^3-1} \times \frac{r^3}{r^{k+2}}\right)$$

$$= \left(\frac{r^{k+1}-1}{r-1}\right) - 2\left(\frac{r^{k+2}-1}{r^3-1}\right)$$

Case (iii) $k \equiv 2 \pmod 3$

$$= \left(r^k + r^{k-2} + r^{k-3} + r^{k-5} + r^{k-8} + r^{k-6} + \dots + r^2 + r^0\right) - \left(r^{k-1} + r^{k-4} + r^{k-7} + \dots + r^4 + r^1\right)$$

$$= r^k\left(1 + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^5} + \dots + \frac{1}{r^k}\right) - \left(r^{k-1} + r^{k-4} + r^{k-7} + \dots + r^4 + r^1\right)$$

$$= r^k\left(1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \dots + \frac{1}{r^k}\right) - 2\frac{r^k}{r}\left(\frac{1}{r^0} + \frac{1}{r^3} + \frac{1}{r^6} + \dots + \frac{1}{r^{k-2}}\right)$$

$$= r^k\left(\frac{1-\left(\frac{1}{r}\right)^{k+1}}{1-\frac{1}{r}}\right) - 2\frac{r^k}{r}\left(\frac{1}{r^0} + \frac{1}{r^3} + \frac{1}{r^6} + \dots + \frac{1}{r^{k-2}}\right)$$

$$= r^k\left(\frac{1-\left(\frac{1}{r}\right)^{k+1}}{1-\frac{1}{r}}\right) - 2\frac{r^k}{r}\left(\frac{1}{r^0} + \frac{1}{r^3} + \frac{1}{r^6} + \dots + \frac{1}{r^{k-2}}\right)$$

$$= r^k\left(\frac{1-\left(\frac{1}{r}\right)^{k+1}}{1-\frac{1}{r}}\right) - 2\frac{r^k}{r}\left(\frac{1-\left(\frac{1}{r^3}\right)^{\frac{k+1}{3}}}{1-\frac{1}{r^3}}\right)$$

$$= \left(\frac{r^{k+1}-1}{r-1}\right) - 2\frac{r^{k+2}}{r^{k+1}}\left(\frac{r^{k+1}-1}{r^3-1}\right)$$

$$= \frac{r^{k+1}-1}{r-1} - 2r\frac{r^{k+1}-1}{(r-1)\left(r^2+r+1\right)}$$

$$= \frac{r^{k+1}-1}{r-1}\left(1 - \frac{2r}{r^2+r+1}\right) = \frac{r^{k+1}-1}{r-1}\left(\frac{r^2+r+1-2r}{r^2+r+1}\right)$$

$$= \frac{r^{k+1}-1}{r-1}\left(\frac{r^2-r+1}{r^2+r+1}\right)$$

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