Numerical Solution of Time Fractional Water Flow Equation and Applications

Dr. Rajkumar Namdevrao Ingle, Sunil Dattatraya Sadegaonkar
1. Principal and H.O.D. Department of Mathematics, Bahirji Smarak Mahavidlaya, Basmathnagar, Dist. Hingoli (M.S.), India.
2. Department of Mathematics, Mahatma Gandhi Mahavidyalaya,Ahmedpur, Latur(M.S.), India.

1.ABSTRACT :

In present paper, we give a detailed analysis for the explicit finite difference approximation for time fractional water flow equation (TFWFE) with initial and boundary conditions. We will also discuss the stability and convergence of the scheme in a bounded domain. Some test problems will be solved to show the application of the scheme. The mathematica software is used to obtain solution graphically.

2.KEYWORDS : Time fractional Water Flow Equation, Stability, Convergence, Mathematica.

3.INTRODUCTION :

Fractional calculus is mere generalization of full integer order integral and differential calculus to real or even complex order. But, it was not used in practice due to its complexity. Fractional differential equations have been used to solve many physical principles in different branches of science and engineering like biology, physics, chemistry, visco elasticity, control systems, thermo dynamics, statistics, finance etc [1]. The numerical techniques are widely used to solve fractional differential equations because of their accuracy and high computational efficiency. Recently, many researchers have worked on finite difference methods for solving the anomalous diffusion equation. The fractional diffusion equations were first studied by Wyss and Schneider [11]. Liu et. Al. has solved the time fractional advection dispersion equation [9]. High order finite difference scheme was used to solve the fractional sub-diffusion equations by Gao and Sun[5]. Graphical flowent method is obtained by Casagrande first time to solve the linear partial differential equation of flow. Two equations like Darcy Equation and Buckingham Equation are combined by Richard's (1931) with the equation of continuity to get an over all relationship. A lot of work is done to understand the physical phenomenon of unsaturated soil. Several methods are developrd by Klute (1972) for obtaining the hydraulic conductivity and diffusivity for unsaturated soils [3]. In this paper we use an effective explicit finite difference scheme for developing the discrete model for time fractional water flow equation which is suitable for simulating random variables whose spatial probability density evolves in time according to this fractional diffusion equation. A flow in aquifer response to sudden change in reservoir level of water is an active part of research. For this the well known water flow equation (Boussinesq's equation) is available in literature to tackle this problem. The hight of the water table u(x, t) in above some reference point is governed by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2};\ \ 0 \le x \le L; t \ge 0$$

We write the fractional water flow equation for aquifer response to sudden change in reservoir level by replacing the first order time derivative by fractional derivative of order α, $0 \le \alpha \le 1$ in the original Boussinesq's equation as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a^2 \frac{\partial^2 u}{\partial x^2}; 0 \le x \le L; t \ge 0$$

To solve a particular model problem of water flow in aquifer response to sudden change in reservoir level, we have to impose proper initial and boundary conditions. For that with an initial water table u(x, t) is at height h_1 for time t = 0, which becomes the initial condition and is mathematically expressed as

$$Initial\ condition: u(x,0) = h_1, 0 \le x \le L$$

Now for left boundary condition, there is a applied source of water placed at height h_1 so as to maintain at all times after t=0 and which is mathematically expressed as

$$u(0,t) = h_1, \quad t \geq 0$$

Now for right boundary condition, there is source of water applied and placed at finite plane x = L so as to maintain at all times after t = 0 the height is h_2, which is mathematically expressed as

$$u(L,t) = h_2, \quad t \geq 0$$

Therefore, the model initial boundary value problem (IBVP) for water flow in aquifer response to sudden change in reservoir level is given as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2 u}{\partial x^2}; 0 \leq x \leq L; t \geq 0$$

subject to the initial and boundary conditions

$$Initial\ condition: u(x,0) = h_1, 0 \leq x \leq L$$

$$Boundary\ conditions: u(0,t) = h_1, u(L,t) = h_2, t \geq 0$$

Where} u(x, t) = volumetric water content of the reservoir, m^3/min,

D = a^2 (m^3/min) is the product of storage coefficient times & the hydraulic coefficient divided by the aquifer thickness,

h_1 = the initial height (m) of water table,

h_2 = the depth of the reservoir on the right at x = L,

x = the distance from one reservoir at higher water level h_1 & to the other reservoir at lower level h_2 t = time (min.), L = width of the strip of land that separates the two reservoirs of depth $h_1$and h_2, (h_2 <$h_1$1).

We consider the Caputo time fractional derivative of order α, 0 ≤ α ≤ 1 and the symmetric second order difference quotient in space at time level t = t_k for solution.

We organize the paper as follows: In section 2 , we develop the explicit fractional order finite difference scheme for time fractional water flow equation. The stability of the solution is proved in section 3 and section 4 deals with convergence of the scheme. The numerical solution of time fractional water flow equation is obtained using Mathematica software in the last section.

4. FINITE DIFFERENCE SCHEME:

We consider the following time fractional water flow equation with initial and boundary conditions

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2 u}{\partial x^2}; 0 \leq x \leq L; t \geq 0 \tag{2.1}$$

$$Initial\ condition: u(x,0) = h_1, 0 \leq x \leq L \tag{2.2}$$

$$Boundary\ conditions: u(0,t) = h_1, u(L,t) = h_2, t \geq 0 \tag{2.3}$$

Where 0 ≤ α ≤ 1, $\frac{\partial^\alpha u}{\partial t^\alpha}$ denotes the time fractional derivative intended in Caputo sense and D = a^2 diffusivity constant. Note that for α = 1, we recover in the limit the well known diffusion equation of Markovian process

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}; \quad 0 \leq x \leq L; \ t \geq 0$$

For the explicit numerical approximation scheme, we define $h = \frac{L}{N}$ and $\tau = \frac{T}{N}$ the space and time steps respectively, such that $t_k = \frac{k}{\tau}$; k = 0,1,...,N be the integration time $0 \le t_k \le T$ and x_i = ih for i = 0,1, ..., N. Define $u_i^k = u(x_i, t_k)$ and let u_i^k denote the numerical approximation to the exact solution $u(x_i, t_k)$.

In the differential equation (2.1), the time fractional derivative term is approximated by the following scheme

$$\frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1}} \frac{1}{(t_{k+1}-\eta)^\alpha} \frac{\partial u(x_i\ \eta)}{\partial \eta} d\eta$$

This can be simplified to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[u_i^{k+1} - u_i^k\right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j \left[u_i^{k+1-j} - u_i^{k-j}\right] \quad (2.1)$$

Where $b_j = [(j+1)^{1-\alpha} - j^{1-\alpha}]$, j= 1, 2, ...,k

For approximating the second order space derivative, we adopt a symmetric second order difference quotient in space at time level t = t_k

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2}$$

Therefore, the fractional approximated equation is

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[u_i^{k+1} - u_i^k\right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j \left[u_i^{k+1-j} - u_i^{k-j}\right] = D\ \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2}$$

After simplification , we get

$u_i^{k+1} = r\ u_{i-1}^k + (1 - 2r - b_1)u_i^k + r\ u_{i+1}^k + \sum_{j=1}^{k-1}[b_j - b_{j+1}]\, u_i^{k-j} + b_k\ u_i^0$, i = 0,1, ..., N, k = 0,1,..., $r = \frac{D\tau^\alpha \Gamma(2-\alpha)}{h^2}$ and $b_j = [(j+1)^{1-\alpha} - j^{1-\alpha}]$

The initial condition is approximated as $u_i^0 = h_1$, i = 1, 2, ... , N.

The boundary conditions are approximated as $u_0^k = h_1$, $u_N^k = h_2$, k = 0, 1, 2, ... , N.

Therefore, the complete fractional approximated IBVP is

$u_i^1 = r\ u_{i-1}^0 + (1 - 2r)u_i^0 + r\ u_{i+1}^0$, For k=0 (2. 4)

$u_i^{k+1} = r\ u_{i-1}^k + (1 - 2r - b_1)u_i^k + r\, u_{i+1}^k + \sum_{j=1}^{k-1}[b_j - b_{j+1}]\, u_i^{k-j} + b_k\, u_i^0$, for k ≥ 1 (2.5)

Initial condition : $u_i^0 = h_1$, i = 1, 2, ... , N. (2.6)

Boundary conditions : $u_0^k = h_1$, $u_N^k = h_2$, k = 0, 1, 2, ... , N. (2.7)

Where $r = \frac{D\tau^\alpha \Gamma(2-\alpha)}{h^2}$ and $b_j = [(j+1)^{1-\alpha} - j^{1-\alpha}]$

Therefore, the fractional approximated IBVP (2.4) to (2.7) can be written in the following matrix equation form

$$\left.\begin{aligned} U^1 &= BU^0 + S, && for\ k = 0 \\ U^{k+1} &= AU^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) U^{k-j} + b_k U^0 + S, && for\ k \geq 1 \end{aligned}\right\}$$

Where $U^K = [U_k^1 \ U_k^2 \ \ldots U_{N-1}^k \ ]^T$, for k = 0,1,2 ..., N and A and B are tri-diagonal matrices .

$$A = \begin{pmatrix} 1-2r-b_1 & r & 0 & 0 & & 0 \\ r & 1-2r-b_1 & 0 & 0 & & 0 \\ & & \ddots & & & \\ & & & 1-2r-b_1 & r & \\ & & & & & \\ & & & & r & 1-2r-b_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1-2r & r & 0 & 0 & & 0 \\ r & 1-2r & 0 & 0 & & 0 \\ & & \ddots & & & \\ & & & 1-2r & r & \\ & & & & & \\ & & & & r & 1-2r \end{pmatrix}$$

and S is a constant column matrix of order N-1 given by

$S = [rh_1, \ 0, 0, \ldots rh_2]^T$,

The above system of algebraic equations is solved by using Mathematica software in section 5.

In the next section, we discuss the stability of the solution of fractional explicit finite difference scheme (2.4) to (2.7) for the time fractional soil moisture diffusion equation (TFSMDE) .

5. STABILITY:

Lemma: The eigenvalues of the N X N tri-diagonal matrix

$$\begin{pmatrix} a & b & \cdots & & \cdots & \cdots & 0 \\ c & a & b & \cdots & & & \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & & \vdots \\ 0 & & & & & \cdots & 0 \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & & \vdots \\ 0 & \cdots & & & & c & a \end{pmatrix}$$

$\lambda_s = a + 2\sqrt{bc}\, cos \frac{s\pi}{N+1}$. s = 1,2,......... N-1

The solution of the explicit finite difference scheme (2.4) to (2.7) for water flow equation (2.1) to (2.3) is stable, when r ≤ min { $\frac{1}{2}$, $\frac{2-b_1}{4}$ }

Proof: We shall use the mathematical induction to analyze the stability.For k=0 and 1 ≤ i ≤ N-1 the eigenvalues of B are given by

$\lambda_s = 1 - 2r + 2r\cos\frac{s\pi}{N} \le 1$; s = 1,2,……… N-1

Again consider

$\lambda_s = 1 - 2r + 2r\cos\frac{s\pi}{N} \ge 1-4r$; s = 1,2,……… N-1

$\lambda_s \ge -1$ when $r \le \frac{1}{2}$, therefore $|\lambda_s| \le 1$ when $r \le \frac{1}{2}$

Hence $\|B\|_2 = \max_{1\le i\le N}|\lambda_s| \le 1$, $\|U^1\|_2 = \|BU^0\|_2$, but $\|B\|_2 \le 1$

Then $\|U^1\|_2 \le \|U^0\|_2$, true for n=1.

We assume $\|U^k\|_2 \le \|U^0\|_2$, n ≤ k is true

We prove that $\|U^{k+1}\|_2 \le \|U^0\|_2$, for n=k+1

For $\|A\|_2$ we have for 1 ≤ i ≤ N-1 the eigen values of A are given by

$\lambda_s = 1 - 2r - b_1 + 2r\cos\frac{s\pi}{N} \le 1 - b_1$; s = 1,2,……… N-1 and $b_1 \ge 0$

$\lambda_s = 1 - 2r - b_1 + 2r\cos\frac{s\pi}{N} \ge 1 - 4r - b_1$; s = 1,2,……… N-1 and $b_1 \ge 0$

$\lambda_s \ge -1$ when $1 - 4r - b_1 \ge -1 \Rightarrow r \le \frac{2-b_1}{4}$

$\Rightarrow |\lambda_s| \le 1$ when $r \le \frac{2-b_1}{4}$ for 1 ≤ i ≤ N-1. Therefore $\|A\|_2 \le 1$.

Hence $\|U^{k+1}\|_2 = \left\| AU^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U^{k-j} + b_k U^0 + S \right\|_2$

$\le (1 - b_1 + b_1 - b_k + b_k)\|U^0\|_2$

$\|U^{k+1}\|_2 \le \|U^0\|_2$

This result is true for n = k+1. Hence by induction $\|U^k\|_2 \le \|U^0\|_2$.

Therefore, this shows that the scheme is stable when r ≤ min { $\frac{1}{2}$, $\frac{2-b_1}{4}$ }. The next section is devoted for convergence of the finite difference scheme.

6. CONVERGENCE:

Let V^k be the vector of exact solution and U^k be the vector of approximate solution of the time fractional water flow equation (1.1) – (1.3) then U^k converges to V^k as (h,τ) → (0,0) when, r ≤ min { $\frac{1}{2}$, $\frac{2-b_1}{4}$ }

Proof: Let $U^K = [u_1, u_2, \dots \dots \dots . . u_{N-1}]^T$,

$V^K = [v_1, v_2, \dots \dots \dots . . v_{N-1}]^T$, Then $E^K = V^K - U^K$

Let us assume that

$$|e_l^k| = \max_{1\le i\le N}|\epsilon_i^k| = \| E^k \|_\infty$$

$T_l^k = \max_{1\le i\le N}|T_i^k|$ = h² o$(\tau + h^2)$, for l = 1, 2, …….

For k=0, from equation (2.4) we have

$u_i^1 = r\ u_{i-1}^0 + (1 - 2r) u_i^0 + r\ u_{i+1}^0$, For k=0

$|e_l^1| = |\ re_{i-1}^0 + (1-2r)e_i^0 + r\ e_{i+1}^0| + r|T_i^1| \le |e_i^0| + rh^2\, o\, (\tau^{1-\alpha} + h^2)$

∴ $\|E^1\|_\infty \le e_i^1\|E^0\|_\infty + \tau^\alpha\Gamma(2-\alpha)\, o\, (\tau^{1-\alpha} + h^2)$, the result holds for n=1.

For n = k, we assume that, $\|E^k\|_\infty \le \|E^0\|_\infty + \tau^\alpha\Gamma(2-\alpha)\, o\, (\tau^{1-\alpha} + h^2)$

∴ For n= k+1, we prove that

$\|E^{k+1}\|_\infty \le \|E^0\|_\infty + (k+1)\tau^\alpha\Gamma(2-\alpha)\, o\, (\tau^{1-\alpha} + h^2)$

Now from equation (2.5) we have

$| E_l^{k+1}| = |\ r\ e_{i-1}^k + 1 - 2r - b_1)e_i^k + r\ e_{i+1}^k + \sum_{j=1}^{k-1}[b_j - b_{j+1}]\, e_i^{k-j} + b_k\ e_i^0\ | + r|T_l^1|$,

$\le |e_l^k| + r|T_l^1| \le \|E^0\|_\infty + r|T_l^1|$

$\le \|E^0\|_\infty + k\tau^\alpha\Gamma(2-\alpha)\, o\, (\tau^{1-\alpha} + h^2) + \tau^\alpha\Gamma(2-\alpha)\, o\, (\tau^{1-\alpha} + h^2)$

$\le \|E^0\|_\infty + (k+1)\tau^\alpha\Gamma(2-\alpha)\, o\, (\tau^{1-\alpha} + h^2)$

Therefore, we conclude that if we assume $r \le \min\{ \frac{1}{2}, \frac{2-b_1}{4} \}$ then $\|E^k\|_\infty \to 0$ as $\tau \to 0$, h→0, which results in the convergence of U_i^k to $U(x_i, t_k)$. Hence the proof is completed.

7. NUMERICAL SOLUTIONS :

In this section, we obtain the approximated solution of time fractional water flow equation with initial and boundary conditions. It is important to use analytical model to obtain the numerical solution of the time fractional water flow equation (TFWFE) by the finite difference scheme. Therefore, we consider the following one-dimensional time fractional water flow equation with initial and boundary conditions to explain the behaviour of fractional diffusion equation by using Mathematica Software.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2 u}{\partial x^2};\ 0 \le x \le 20,\ 0 \le \alpha \le 1,\ t \ge 0$$

$$Initial\ condition: u(x,0) = 0.2, \quad 0 \le x \le 20$$

$$Boundary\ conditions: \quad u(0,t) = 0.2, \quad u(20,t) = 0.1, \quad t \ge 0$$

with the diffusion coefficient D = 10. The numerical solutions are obtained at t = 0.3 by considering the parameters τ = 0.03, h = 2, α = 0.9, α = 0.8 is simulated in the following figure.

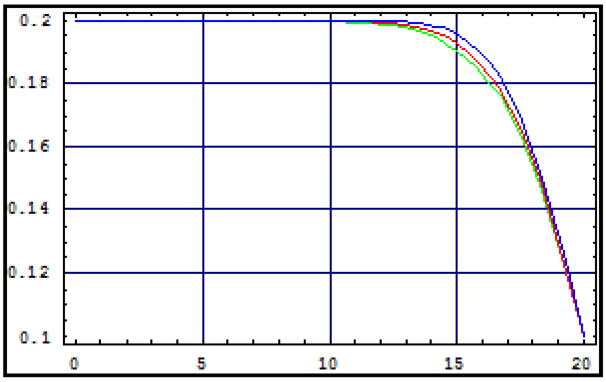

Fig.5.1: The water flow profile with t = 0.3, h = 2, α = 1 (blue), α = 0.9(red), α = 0.8 (green)

Following table shows the comparison of exact solution and numerical solutions.

X	$U_{EXACT}(1)$	$U_{\alpha=0.9}(2)$	$U_{\alpha=0.8}$ (3)	(1) – (2)	(1) – (3)
2	0.2000	0.2000	0.2000	0.0	0.0
4	0.2000	0.200001	0.199999	0.00001	.000001
6	0.2000	0.20000	0.199996	0.0	.000004
8	0.2000	0.199992	0.199968	0.000008	.000032
10	0.199996	0.199916	0.199785	0.00008	.000211
12	0.199891	0.199369	0.198842	0.00052	.001049
14	0.198569	0.196508	0.195072	0.002061	.003497
16	0.189753	0.185585	0.183243	0.004168	.00651
18	0.158578	0.15578	0.154284	0.002798	.004294

Table 5.1:Comparison of exact and numerical solutions at t = 0.3

8.CONCLUSIONS:

1) The proposed explicit difference approximation for time fractional soil moisture diffusion equation can be reliably applied to solve any fractional order dynamical systems and controllers.

2) The stability and convergence of the scheme is also proved effectively.

3) The comparison of exact and numerical solutions shows that results are compatible with theoretical analysis.

9.REFERENCES :

(1)O. P. Agrawal, Solution for a Fractional Diffusion-Wave Equation Defined in a Bounded Domain, J. Nonlinear Dynamics 29, (2002).

(2)Daniel Hillel, Introduction to Soil Physics, Academic Press (1982).

(3) Don Kirkham and W.L. Powers, Advanced Soil Physics Wiley-Interscience (1971).

(4)G. Gao and Z. Sun, A compact finite difference scheme for the fractional sub-diffusion equations, J. Comput. Phys., 230:586-595, (2011).

(5) R. Gorenflo and F. Mainardi, Approximation of Levy-Feller Diffusion by Random Walk, Journal for Analysis and its Applications (ZAA) 18:231-246,(1999).

(6) Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comput. Phys. 225 , pp.1533-1552: (2007)

(7) Pater A.C., Raats and Martinus TH. Ven Genuchten, Milestones in Soil Physics, J. Soil Science 171,1(2006).

(8) I. Podlubny, Fractional Differential equations, Academic Press, San Diago (1999).

(9) S . Shen, F.Liu, Error Analysis of an explicit Finite Difference Approximation for the Space Fractional Diffusion equation with insulated ends, ANZIAM J.46 (E), { pp. C871 - C887}: (2005).

(10) G.D. Smith, Numerical Solution of Partial Differential Equations,(2ndEdn.), Clarendon Press, Oxford, (1978).

(11) W. Wyss and W. R. Schneider, Fractional Diffusion and wave equations, J. Math. Phys. 30, (1989).

(12) D.B. Dhaigude and K. C. Takale, A Two Level Explicit Finite Difference Scheme for One-Dimensional Penne's Bioheat Equation, Recent Advances in Mathematical Sciences and Applications, p. 230-238 (2009).

(13) D.B. Dhaigude, K. C. and Takale , A Weighted Average Finite Difference Scheme for One-Dimensional Penne's Bioheat Equation, Neural, Parallel and Scientific Computations, USA, p. 230-238, 4(2010).

(14) K. C. Takale, D.B. Dhaigude and V.R. Nikam, Douglas Higher Order Finite Difference Scheme for One-Dimensional Penne's Bioheat Equation, Int. Jr. of Advanced Engineering and Applications, p. 61-65 (2011).

(15) S.M.Jogdand, K.C.Takale and V.C.Borkar, Fractional Order Finite Difference Scheme For Soil Moisture Diffusion Equation and its Applications, IOSR Journal of Mathematics (IOSR-JM)}, Volume 5, pp. 12-18, 4 (2013).

(16)Manisha P. Datar, Sharvari R. Kulkarni and Kalyanrao C. Takale, Finite Difference Approximation for Black-Scholes Partial Differential Equation and Its Application, Int Jr. of Mathematical Sciences and Applications, Vol. 3, No.1, (2013).