*-EPIMORPHISMS IN PSEUDO-COMPLEMNTED ALMOST SEMILATTICES

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Abstract : The concept of *-epimorphism between Pseudo-complemented Almost Semilattices (PCASLs) L and M is introduced and proved that if $f: L \to M$ is a *-homomorphism, then $ker(f)$ is a kernel ideal of L. It is proved that $f: L \to M$ is a *-epimorphism, then the mappings $\overrightarrow{f}$: P(L) →P(M) and $\overleftarrow{f}$: P(M)→ $P(L)$ preserves kernel ideals. If L and M are *-commutative PCASLs in which $x \leq x^{**}$, for all x and if $f: L \to M$ is *-epimorphism then established a lattice epimorphism $\overrightarrow{f}_k$, between complete implicative lattices $KI(L)$ and $KI(M)$ proved that $\overrightarrow{f}_k$ is dually range closed. It is proved that complete lattices $KI(L)$ and $I(S(L))$ of all ideals in the Boolean algebra $S(L)$ of all *-elements in *-commutative PCASL L are isomorphic. The concept of co-kernel of a *-congruence on PCASL L is introduced and proved that if L is a *-commutative PCASL in which $x \leq x^{**}$ for all $x \in L$ and K is a filter of L, then the relation S_K on L defined by $(x, y) \in S_K$ if and only if $x \circ t = y \circ t$, for some $t \in K$ is the smallest *-congruence with co-kernel K. Also, introduced the concept of *-filter in PCASL L and proved that if K is *-filter of *-commutative PCASL L then the *-congruence $S_K \vee \psi$ is the largest *-congruence with co-kernel K, where ψ is a relation on L defined by $(x, y) \in \psi$ if and only if $x^{**} = y^{**}$ which is a *-congruence.

Key words: *-homomorphism, *-epimorphism, Kernel Ideals, Complete Implicative Lattice, Co-kernel,*-filter, *-congruence, Smallest *-congruence, Largest *-congruence.

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1. INTRODUCTION

The concept of Pseudo-complemented Almost Semilattices(PCASL) was introduced by, Nanaji Rao, G. and Sujatha Kumari, S [4]. They proved several basic properties of pseudo-complementation * on L and proved that the pseudo-complementation * on an ASL L is equationally definable. They proved that the set of all *-elements in a *-commutative PCASL form a Boolean algebra which is independent(up to isomorphism) of the pseudo-complementation *. Next, the concepts of kernel ideal, *-ideal and *-congruence in *-commutative PCASL L were introduced by Nanaji Rao, G. and Sujatha Kumari,S [5], they derived a necessary and sufficient conditions for an ideal in *-commutative PCASL L to become a kernel ideal, established the smallest *-congruence with given kernel ideal , largest *-congruence with given kernel ideal and characterized the largest *-congruence in terms of smallest *-congruence and the *-congruence ψ defined on L by $(x, y) \in \psi$ if and only if $x^{**} = y^{**}$. In [6], Nanaji Rao,G. and Sujatha Kumari,S, proved some basic properties of ideal quotient in ASL L and also proved that the set $I^*(L)$ of all *-ideals of *-commutative PCASL L is complete lattice with respect to set inclusion. Next, they proved that the centre of $I^*(L)$ is trivial and proved that the set $KI(L)$ of all kernel ideals in *-commutative PCASL L in which $x \leq x^{**}$ for all $x \in L$ is complete implicative lattice.

In this paper, we introduced the concept of *-epimorphism between PCASLs L and M and proved that if $f: L \to M$ is a *-homomorphism, then $ker(f)$ is a kernel ideal of L. Moreover we proved that if $f: L \to M$ is a*-epimorphism, then the mappings $\overrightarrow{f}$: P(L) →P(M) and $\overleftarrow{f}$: P(M)→ $P(L)$ preserves kernel ideals. Also, proved that if L and M are *-commutative PCASLs in which $x \leq x^{**}$ for all x and if

$f: L \to M$ is *-epimorphism then the mapping $\overrightarrow{f}_k : KI(L) \to KI(M)$ is a lattice epimorphism and also proved that $\overrightarrow{f}_k$ is dually range closed. Next, we proved that complete lattices $KI(L)$ of all kernel ideals of *-commutative PCASL L in which $x \leq x^{**}$ for all $x \in L$ and $I(S(L))$ of all ideals of the Boolean algebra of all *-elements in *-commutative PCASL L are isomorphic. Again, we introduced the concept of co-kernel of a *-congruence on PCASL L and proved that if L is *-commutative PCASL in which $x \leq x^{**}$, for all $x \in L$ and K is a filter of L, then the relation S_K on L defined by $(x,y) \in S_K$ if and only if $x \circ t = y \circ t$, for some $t \in K$ is the smallest *-congruence with co-kernel K. Also, we introduced the concept of *-filter in PCASL L and proved the set $F^*(L)$ of all *-filters of PCASL L , is a complete implicative lattice. Finally, we proved that if K is a *-filter of *-commutative PCASL L then the *-congruence $S_K \vee \psi$ is the largest *-congruence with co-kernel K, where ψ is a relation on L defined by $(x,y) \in \psi$ if and only if $x^{**} = y^{**}$which is a *-congruence.

2. PRELIMINARIES

In this section we collect few important definitions and results which are already known and which will be used more frequently in the text.

2.1. Definition: An almost semilattice(ASL) is an algebra $(L,\circ)$ where L is a non-empty set and $\circ$ is a binary operation on L, satisfing the following conditions:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)

(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)

(3) $x \circ x = x$, for all $x, y, z \in L$. (Idempotent Law)

2.2. Definition: An ASL with 0 is an algebra $(L,\circ,0)$ of type $(2,0)$ satisfing the following conditions:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)

(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)

(3) $x \circ x = x$ (Idempotent Law)

(4) $0 \circ x = 0$, for all $x, y, z \in L$.

2.3. **Definition:** *Let L be a non-empty set. Define a binary operation $\circ$ on L by $x \circ y = y$, for all $x, y \in L$. Then $(L,\circ)$ is an ASL and is called discrete ASL.*

2.4. Theorem: *Let $(L,\circ)$ be an ASL. Define a relation $\leq$ on L by $a \leq b$ if and only if $a \circ b = a$. Then $\leq$ is a partial ordering on L.*

2.5. Theorem: *Let $(L,\circ)$ be an ASL. Then for any $a, b \in L$ with $a \leq b$ we have $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$, for all $c \in L$.*

2.6. Theorem: Let $(L,\circ)$ be an ASL. Then for any $a, b \in L$, we have the following:

(1) $a \circ b \leq b$.

(2) $a \circ b = b \circ a$ whenever $a \leq b$.

If $(L,\circ)$ *is an ASL* then by an ideal of L is a non-empty subset I of L which satisfies $x \circ t \in I$ for any $x \in I$ and $t \in L$. It can be easily verified, for any a in an ASL L, $(a] = \{a \circ x : x \in L$ is an ideal of L and called principal ideal generated by a.

2.7.Definition: A non-empty subset F of an $ASL\ L$ is said to be a filter if F satisfing the following conditions :

(1) $x, y \in F$ implies $x \circ y \in F$,

(2) If $x \in F$ and $a \in L$ such that $a \circ x = x$ then $a \in F$.

2.8.Definition: *Let* $(L,\circ)$ *be an ASL. Then an element* $m \in L$ *is said to be unimaximal if* $m \circ x = x$, *for all* $x \in L$.

2.9.Corollary: Let L be an ASL and I be an ideal of L. Then, for any $a,b \in L, a \circ b \in I$ if and only if $\mathrm{b} \circ \mathrm{a} \in \mathrm{I}$.

2.10.Lemma: Let L be an ASL and $a,b \in L$, Then $a \in (b]$ if and only if $a = b \circ a$.

2.11.**Theorem:** Let $(L,\circ)$ be an ASL with 0. Then for any $a,b \in L$, we have the following:

(1) $a \circ 0 = 0$.
(2) $a \circ b = 0$ if and only if $b \circ a = 0$.
(3) $a \circ b = b \circ a$ whenever $a \circ b = 0$.

2.12. Definition: For any non-empty subset A of an ASL L with 0, define $A^* = \{x \in L : x \circ a = 0,$ for all $a \in A\}$. Then A^* is called the annihilator of A. It can be easily seen that A^* is an ideal of L. Also, note that, if $A = \{a\}$, then we denote $A^* = \{a\}^*$ by $[a]^*$.

2.13. Theorem: Let L be an ASL with 0. Then a unary operation $* : L \to L$ is a pseudo-complementation on L if and only if it satisfies the following conditions:

(1) $a^* \circ b = (a \circ b)^* \circ b$
(2) $0^* \circ a = a$
(3) $0^{**} = 0$.

2.14. Definition: Let L and L' be two $ASLs$ with zero elements 0 and $0'$ respectively. Then a mapping $f : L \to L'$ is called an ASL homomorphism if it satisfies the following conditions :

(1) $f(a \circ b) = f(a) \circ f(b)$, for all $a, b \in L$
(2) $f(0) = 0'$.

2.15. Definition: Let L be an ASL with zero. Then a unary operation $a \mapsto a^*$ on L is said to be pseudo-complementation on L if, for any $a,b \in L$, it satisfies the following conditions:

(1) $a \circ b = 0 \Rightarrow a^* \circ b = b$
(2) $a \circ a^* = 0$.

2.16. Lemma: Let L be a $PCASL$. Then for any $a,b \in L$, we have the following:

(1) $0^* \circ a = a$
(2) 0^* is unimaximal
(3) $a^{**} \circ a = a$
(4) a is unimaximal $\Rightarrow$ $a^* = 0$
(5) $0^{**} = 0$.

2.17. Definition: An ideal I of a $PCASL$ L is said to be a kernel ideal if I is the kernel of a *-congruence on L.

Remark: Whether *-elements commutes are not, is not known so far in pseudo-complemented ASL with pseudo-complementation *, investigation is goin on.

Here onwords by a $*$-commutative PCASL L we mean L is a PCASL with pseudo-complementation * in which all $*$-elements are commute.

When $(L,\circ)$ is a *-commutative PCASL then an ideal I of L is a kernel ideal if and only if for any $x, y \in I, (x^* \circ y^*)^* \in I$ ([5], Theorem 3.12).

2.18. Theorem: Let L be a $*$-commutative $PCASL$. Then for any $a, b \in L$, we have the following

(1) $a \le b \Rightarrow b^* \le a^*$

(2) $a^{***} = a^*$

(3) $a^* \le b^* \Leftrightarrow b^{**} \le a^{**}$.

2.19. **Theorem:** Let L be a $*$-commutative $PCASL$. Then for any $a, b \in L$, we have the following:

(1) $(a \circ b)^{**} = a^{**} \circ b^{**}$

(2) $(a \circ b)^* = (b \circ a)^*$

(3) $a^*, b^* \le (a \circ b)^*$.

2.20. Definition: Let L be a $PCASL$ with pseudo-complementation *. Then a congruence relation θ on L is said to be a $*$-congruence if for any $(x, y) \in \theta, (x^*, y^*) \in \theta$.

2.21. Theorem: Let L be a $*$-commutative $PCASL$ and let θ be a congruence on L. Then θ is a $*$-congruence if and only if for any $(x, 0) \in \theta$ implies $(x^*, 0^*) \in \theta$.

2.22. Theorem: Let L be a *-commutative PCASL in which $x \le x^{**}$ for all $, x \in L$. Then order by set inclusion, $KI(L)$ forms a complete implicative lattice in which the operations are as follows: If $\{I_\alpha : \alpha \in \Delta\}$ is any family of kernel ideals of L, $\wedge_{\alpha\in\Delta} I_\alpha = inf_{KI(L)}\{I_\alpha : \alpha \in \Delta\} = \cap_{\alpha\in\Delta} I_\alpha$, $\vee_{\alpha\in\Delta} I_\alpha = sup_{KI(L)}\{I_\alpha : \alpha \in \Delta\} = \{x \in L : (\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \triangle)(\exists x_i \in I_{\alpha_i})(x \le (\circ_{i=1}^{n}\ x_i^*)^*)\}$ and residuals in $KI(L)$ coinsides with the corresponding residuals in $I^*(L)$.

2.23. Theorem: Let L be a *-commutative PCASL in which $x \le x^{**}$, for all $x \in L$. Then the following conditions are equivalent:

(1) Every ideal of L is a kernel ideal.

(2) Every principal ideal of L is a kernel ideal.

(3) L is a Boolean algebra.

2.24. Theorem: If P is a partly ordered set bounded above each of whose non-void subsets R has an infimum, then each non-void subset of P will have a supremum, too, and by the definitions $\cap R = \inf R$, $\cup R = \sup R$, then P becomes a complete lattice.

2.25. Corollary: If a bounded lattice is complete with respect to one of the lattice operations, it is also complete with respect to the other.

2.26. Definition: If we are given a set A, a mapping C : Su(A)→ Su(A) is called a closure operator on A if, for X, Y ⊆ P(A), it satisfies:

(1) X ⊆ C(X)
(2) $C^2(X) = C(X)$
(3) X ⊆ Y implies C(X) ⊆ C(Y).

A subset X of A is called closed subset if C(X) = X. The poset of closed subsets of A with set inclusion as the partial ordering is denoted by L_C .

2.27. Theorem: Let C be a closure operator on a set A. Then L_C is a complete lattice with $\underset{i\in I}{\wedge} C(A_i) = \underset{i\in I}{\cap} C(A_i)$ and $\underset{i\in I}{\vee} C(A_i) = C(\underset{i\in I}{\cup} A_i)$

2.28. Definition: If L, M are partially orderd sets a map $f: L \to M$ is residual if and only if f is isotone and there exists a unique isotone map $f^+: M \to L$ such that $f^+ \circ f \geq id_L$ and $f \circ f^+ \leq id_M$. The unique map f^+ is called the residual of f.

3. *-epimorphisms

In this section we introduce the concept of *-epimorphism and prove that if $f: L \to M$ is a *-homomorphism, then $ker(f)$ is a kernel ideal of L. More over we prove that if $f: L \to M$ is a *-epimorphism then the mapping $\overrightarrow{f}$: P(L) →P(M) and $\overleftarrow{f}$: P(M) → $P(L)$ preserves kernel ideals. Also, prove that if L and M are *-commutative PCASLs in which $x \leq x^{**}$ for all x and if $f: L \to M$ is *-epimorphism then the $\overrightarrow{f}_k : KI(L) \to KI(M)$ is a lattice epimorphism and also prove that $\overrightarrow{f}_k$ is dually range closed. We prove that complete lattices $KI(L)$ and $I(S(L))$ are isomorphic. We give a necessary and sufficient condition the map $\overrightarrow{f}_k$ is a *-epimorphism. First, we begin this section with the following definition.

3.1. **Definition:** Let L, M be pseudo-complemented almost semilattices. A homomorphism $f: L \to M$ is said to be *-epimorphism if f is onto and $f(x^*) = (f(x))^*$, for all $x \in L$.

In the following we prove that the kernel of a *-homomorphism is a kernel ideal.

3.2. **Theorem:** Let L, M be PCASLs. If $f: L \to M$ is a *-homomorphism. Then $ker(f) = \{x \in L : f(x) = 0\}$ is a kernel ideal of L.

Proof : Suppose $f: L \to M$ is a *-homomorphism. Since $f(0) = 0, 0 \in$ ker(f). Therefore ker(f) is non-empty subset of L. Let $x \in$ ker(f) and $a \in L$. Then $f(x) = 0$. Now, Consider $f(x \circ a) = f(x) \circ f(a) = 0 \circ f(a) = 0$. Therefore $x \circ a \in$ ker(f). Hence ker(f) is an ideal of L. Let $x, y \in$ ker(f). Then $f(x) = 0, f(y) = 0$. Now, consider $f((x^* \circ y^*)^*) = (f(x^* \circ y^*))^* = \left(f(x^*) \circ f(y^*)\right)^* = ((f(x))^* \circ (f(y))^*)^* = (0^* \circ 0^*)^* = 0^{**} = 0$. Hence $(x^* \circ y^*)^* \in$ ker(f). Thus ker(f) is an ideal of L.

Now, we shall introduce the following notation. Given a mapping $f: E \to F$, we shall denote by $\overrightarrow{f}$: P (E) → $P(F)$ and $\overleftarrow{f}$: P (F) → $P(E)$, the induced mappings given by the prescriptions

$$(\forall X \subseteqq E)\overrightarrow{f}(X) = \{f(x): x \in X\} = f(X),$$
$$(\forall Y \subseteqq F)\overleftarrow{f}(Y) = \{x \in E : f(x) \in Y\} = f^{-1}(Y).$$

Next, we prove that if $f: L \to M$ is a *-epimorphism, then $\overrightarrow{f}$ and $\overleftarrow{f}$ preserve kernel ideal.

3.3. Theorem: Let L, M be PCASLs and let $f: L \to M$ be a *-epimorphism. Then $\overrightarrow{f}$ and $\overleftarrow{f}$ preserve kernel ideal.

Proof: Suppose $f: L \to M$ is a *-epimorphism. First, we shall prove that if I is a kernel ideal of L, then $\overrightarrow{f}(I)$ is a kernel ideal of M. Let I be a kernel ideal of L. Since $0 \in I, f(0) \in \overrightarrow{f}(I)$. Therefore $\overrightarrow{f}(I)$ is non-empty subset of M. Let $f(x) \in f(I)$ and $y \in M$. Since f is onto, there exists $z \in L$ such that $f(z) = y$. Now, since $x \in I, z \in L$, $x \circ z \in I$. This implies $f(x \circ z) \in f(I)$. Therefore $f(x) \circ y = f(x) \circ f(z) = f(x \circ z) \in f(I)$. Hence $f(I)$ is an ideal of M. Let $f(x), f(y) \in f(I)$. Then $x, y \in I$. Since I is a kernel ideal, $(x^* \circ y^*)^* \in I$. This implies $f(x^* \circ y^*)^* \in f(I)$. It follows that $(f(x)^* \circ f(y)^*)^* \in f(I)$. Hence $f(I)$ is a kernel ideal of M. Thus $\overrightarrow{f}(I)$ is a kernel ideal of M. Suppose J is a kernel ideal of M. Since $0 = f(0) \in J, 0 \in f^{-1}$(J). Therefore f^{-1}(J) is a non-empty subset of L. Let $x \in f^{-1}$(J) and $a \in L$. This implies $f(x) \in J$ and $f(a) \in f(L) \subseteq M$. It follows that $f(x) \circ f(a) \in J$ Therefore $f(x \circ a) \in J$. Hence $x \circ a \in f^{-1}$(J). Thus f^{-1}(J) is an ideal of L. Let $x, y \in f^{-1}$(J). Then $f(x), f(y) \in J$. It follows that $((f(x))^* \circ (f(y))^*)^* \in J$. This implies $(f(x^*) \circ f(y^*))^* \in J$. Therefore $(f(x^* \circ y^*))^* \in J$. It follows that $f((x^* \circ y^*)^*) \in J$. Hence $(x^* \circ y^*)^* \in f^{-1}$(J). Thus f^{-1}(J) is a kernel ideal of L. Therefore $\overleftarrow{f}(J)$ is a kernel ideal of L.

Note that if L and M are *-commutative PCASLs in which $x \le x^{**}$, for all x and if $f: L \to M$ is a *-epimorphism, then in view of theorem 3.3, f is induces a surjective reisduated mapping $\overrightarrow{f}_k : KI(L) \to KI(M)$ described by $I \mapsto \overrightarrow{f}(I)$ the residual of this being the injective mapping $\overleftarrow{f}_k : KI(M) \to KI(L)$ described by $J \mapsto \overleftarrow{f}(J)$. Now, we prove the following.

Theorem 3.4. Let L, M be a *-commutative PCASLs in which $x \le x^{**}$, for all x and let $f: L \to M$ is a *-epimorphism. Then $\overrightarrow{f}_k$ is a lattice epimorphism.

Proof. Suppose $f: L \to M$ is a *-epimorphism. Now, we have $\overrightarrow{f}_k : KI(L)$→KI(M) defined by $\overrightarrow{f}_k(I) = \overrightarrow{f}$ (I) = $\{f(x): x \in I\}$, for all $I \in KI(L)$. Let $I, J \in KI(L)$. Then clearly, $\overrightarrow{f}_k(I \cap J) = \overrightarrow{f}_k(I) \cap \overrightarrow{f}_k(J)$. Now, we shall prove that $\overrightarrow{f}_k(I \vee J) = \overrightarrow{f}_k(I) \vee \overrightarrow{f}_k(J)$. That is enough to prove that $f(I \vee J) = f(I) \vee f(J)$. Let $f(x) \in f(I \vee J)$. Then $x \in I \vee J$. This implies $x \le (a^* \circ b^*)^*$, where $a \in I, b \in J$. Hence $f(a) \in f(I)$ and $f(b) \in f(J)$. It follows that $(f(a)^* \circ f(b)^*)^* \in f(I) \vee f(J)$ and hence $f((a^* \circ b^*)^*) \in f(I) \vee f(J)$. But, we have $f(x) \le f((a^* \circ b^*)^*)$. Hence $f(x) \in f(I) \vee f(J)$. Thus $f(I \vee J) \subseteq f(I) \vee f(J)$. Conversely, suppose $y \in f(I) \vee f(J)$. Since f is onto, there exists $x \in L$ such that $f(x) = y$. Now, $f(x) \in f(I) \vee f(J)$. Then $f(x) \le ((f(a))^* \circ (f(b))^*)^*$, for some $a \in I$ and $b \in J$. This implies $f(x) \le f((a^* \circ b^*)^*)$, where $a \in I$ and $b \in J$. But, we have $(a^* \circ b^*)^* \in I \vee J$. Therefore $f((a^* \circ b^*)^*) \in f(I \vee J)$. It follows that $f(x) \in f(I \vee J)$. Hence $y \in f(I \vee J)$. Therefore $f(I) \vee f(J) \subseteq f(I \vee J)$. Thus $f(I) \vee f(J) = f(I \vee J)$. Therefore $\overrightarrow{f}_k$ is a homomorphism. Let $J \in$ KI(M). Then we have f^{-1}(J) is a kernel ideal of L. Therefore $f^{-1}(J) \in KI(L)$. Now, we shall prove that $f(f^{-1}$(J)) = J. Let $y \in J$. Since f is onto, there exists $x \in L$ such that $f(x) = y$. Therefore $f(x) \in J$. This implies $x \in f^{-1}$(J). Hence $y = f(x) \in f(f^{-1}$(J)). Therefore $J \subseteq f(f^{-1}$(J)). Clearly, $f(f^{-1}$(J)) $\subseteq J$. Thus $f(f^{-1}$(J)) = J. It follows that $\overrightarrow{f}_k(f^{-1}$(J)) = J. Thus $\overrightarrow{f}_k$ is onto and hence is an epimorphism.

It can be easily seen that if f is a *-epimorphism, then the induced residuated mapping is surjective and so is range closed. In the following, we prove the residuated mapping $\overrightarrow{f}_k$ is dually range closed.

3.5. Theorem: If $f: L \to M$ is a *-epimorphism, then $\overrightarrow{f}_k : KI(L) \to KI(M)$ is dually range closed.

Proof. Now, we shall prove that $\overrightarrow{f}_k$ is dually range closed. That is enough to prove , for every $I \in KI(L)$, $\overleftarrow{f}_k(\overrightarrow{f}_k(I)) = sup_{KI(L)}\{I, \ker(f)\}$. Let $I \in KI(L)$. Since $\overrightarrow{f}$, $\overleftarrow{f}$ preserves kernel ideals, it follows that $\overleftarrow{f}_k(\overrightarrow{f}_k(I)) \in KI(L)$. Then clearly, $\overleftarrow{f}_k(\overrightarrow{f}_k(I))$ is an upper bound of $\{I, \ker(f)\}$. Let $x \in \overleftarrow{f}_k(\overrightarrow{f}_k(I))$.

Then we have $f(x) = f(i)$, for some $i \in I$. It follows that $f(x) \le f(i^{**})$. Hence we get $f(x) \le (f(i^*)^*)$. Let $H \in KI(L)$ such that H is an upper bound of $\{I, \ker(f)\}$. Then we have $I, \ker(f) \subseteq H$. Now, we shall prove that $\overleftarrow{f}_k(\overrightarrow{f}_k(I)) \subseteq H$. Let $x \in \overleftarrow{f}_k(\overrightarrow{f}_k(I))$. Then we have $f(x) \in f(I)$. Hence we can write $f(x) = f(i)$, for some $i \in I$. Now, since $i \le i^{**}$, $f(i) \le f(i^{**}) = f(i^*)^* = (f(i^*))^*$.
Now, consider $f(x \circ i^*) = f(x) \circ f(i^*) \le \left(f(i^*)\right)^* \circ f(i^*)$. It follows that $f(x \circ i^*) = 0$. Hence $x \circ i^* \in \ker(f)$. Therefore $i, x \circ i^* \in H$. We have $x \le x^{**} \le (i^* \circ x^*)^*$. But $(i^* \circ x^*)^* = (x^* \circ i^*)^* = ((x \circ i^*)^* \circ i^*)^* = (i^* \circ (x \circ i^*)^*)^*$. It follows that $x \le (i^* \circ (x \circ i^*)^*)^*$. Since H is a kernel ideal, $x \in H$. Thus $\overleftarrow{f}_k(\overrightarrow{f}_k(I))$ is the $sup_{KI(L)}\{I, \ker(f)\}$.

3.6. Corollary: Let L be a *-commutative PCASL in which $x \le x^{**}$ for all $, x \in L$. If $KI_f(L)$ is the set of all kernel ideals of L contains $\ker(f)$, then $KI_f(L) \cong KI(L)$.
Proof. Proof follows by theorem 3.5.

Recall that if L is a *-commutative PCASL, then the set $S(L) = \{a^{**} : a \in L\}$ is a Boolean algebra with the original determination of the meet operation $a \circ b$ and of the order relation $a \le b$, the Boolean complement of an element being its pseudo-complement for these element, the Boolean join operation is given by the formula $a \underline{\vee} b = (a^* \circ b^*)^*$. It can be easily seen that I is an ideal of a Boolean algebra B if and only if I is a kernel ideal of B. Now, we prove that $KI(L)$ is isomorphic with $IS(L)$. For, this first we need the following.

3.7. Lemma: Let L be a *-commutative PCASL. Define $g: L \to S(L)$ by $g(a) = a^{**}$, for all $a \in L$. Then g is a *- epimorphism.
Proof. Clearly, g is well defined. Let $a, b \in L$. Then $g(a \circ b) = (a \circ b)^{**} = a^{**} \circ b^{**} = g(a) \circ g(b)$ and $g(a^*) = a^{***} = (a^{**})^* = (g(a))^*$. It follows that g is a *-homomorphism. Now, let $a \in S(L)$. Then we have $a = a^{**}$ and $g(a) = a^{**} = a$. Thus g is a *-epimorphism.

Recall that if B is a Boolean algebra, then I is an ideal of B if and only if then I is a kernel ideal of B. It follows that $I\big(S(L)\big) = KI(S(L))$.

3.8. Theorem: $KI(L) \cong I(S(L))$.
Proof. By lemma 3.7. $g: L \to S(L)$ is a *-epimorphism. Therefore by theorem 3.4. $\overrightarrow{g}_k: KI(L) \to KI(S(L))$ is a lattice epimorphism. It follows that $\overrightarrow{g}_k: KI(L) \to I(S(L))$ is a lattice epimorphism. Now, cosider

$$\begin{aligned} \mathrm{Ker}(g) &= \{x \in L: g(x) = 0\} \\ &= \{x \in L: x^{**} = 0\} \\ &= \{x \in L: x = 0\} \\ &= \{0\} \end{aligned}$$

Therefore $KI(L)$ is isomorphic to $I(S(L))$.

Recall that if B is a complete Boolean algebra, then the set of all ideals in B form a Stone lattice. [3]

3.9. Corollary: If $S(L)$ is complete, then $KI(L)$ is Stone lattice.

In the following, we give necessary and sufficient condition that the induced residuated mapping $\overrightarrow{f}_k$ is a *-epimorphism.

3.10. Theorem: If $f: L \to M$ is a *-epimorphism, then the following statements are equivalent:
(1) $\overrightarrow{f}_k : KI(L) \to KI(M)$ is a *-epimorphism
(2) $\ker(f)$ is a principal ideal.
Proof. (1)⇒(2): Suppose $\overrightarrow{f}_k : KI(L) \to KI(M)$ is a *-epimorphism. Put $A = \ker(f)$. Then we have $\ker(f)$ is a kernel ideal of L and hence $f(A)$ is a kernel ideal of M. It follows that $(f(A))^*$ is a kernel ideal of M.

Therefore $(f(A))^* \subseteq M$. Conversely, let $m \in M$ and $f(a) \in f(A)$. Then $a \in A = \ker(f)$. This implies $f(a) = 0$. It follows that $m \circ f(a) = 0$. Therefore $m \in (f(A))^*$. Hence $M \subseteq (f(A))^*$. Thus $(f(A))^* = M$. Now, $\vec{f}_k(A \vee A^*) = \vec{f}_k(A^*) = (\vec{f}_k(A))^* = (f(A))^* = M$. On the other hand $\vec{f}_k(L) = f(L) = M$. It follows that $A \vee A^* = L$. But, we have $A \cap A^* = \{0\}$. Therefore A is complemented. Hence by theorem 2.23, A is principal ideal of L. Thus $\ker(f)$ is a principal ideal of L. (2) ⇒(1): Suppose $\ker(f)$ is a principal ideal of L. That is we can write $\ker(f) = (a]$, for some $a \in L$. But, by theorem 3.4. $\vec{f}_k$ is a lattice epimorphism. It is enough to prove that $\vec{f}_k(I^*) = (\vec{f}_k(I))^*$, for all $I \in KI(L)$. Let $x \in \vec{f}_k(I^*) = f(I^*)$. This implies $x = f(y)$, for some $y \in I^*$. Therefore $y \circ i = 0$, for all $i \in I$. It follows that $f(y \circ i) = f(0) = 0$, for all $i \in I$. This implies $f(y) \circ f(i) = 0$, for all $i \in I$. Hence $x \circ f(i) = 0$, for all $i \in I$. Therefore

$x \in (\vec{f}_k(I))^*$. Hence $\vec{f}_k(I^*) \subseteq (\vec{f}_k(I))^*$. Conversely, let $x \in (\vec{f}_k(I))^*$. This implies $x \circ f(i) = 0$, for all $i \in I$. Since f is onto there exists $z \in L$ such that $f(z) = x$. It follows that $f(z) \circ f(i) = 0$, for all $i \in I$. Hence $f(z \circ i) = 0$, for all $i \in I$. Therefore $z \circ i \in \ker(f) = (a]$, for all $i \in I$. It follows that $z \circ i = a \circ z \circ i$, for all $i \in I$. This implies $a^* \circ z \circ i = 0$, for all $i \in I$. Therefore $a^* \circ z \in I^*$. This implies $f(a^* \circ z) \in f(I^*)$. Now, consider $f(a^* \circ z) = f(a^*) \circ f(z) = f(a)^* \circ x = 0^* \circ x$ (since$a \in (a] = \ker(f), f(a) = 0$) $= x$

(since 0^* is unimaximal). Therefore $x \in f(I^*)$. Hence $(\vec{f}_k(I))^* \subseteq \vec{f}_k(I^*)$. Thus $\vec{f}_k(I^*) = (\vec{f}_k(I))^*$. Therefore $\vec{f}_k$ is a *-epimorphism.

4.*-filters

In this section we observe that if F is a filter of PCASL L and $x^{**} \in F$ need not imply $x \in F$ by means of example. This motivate us to introduce the concepts of *-filter and co-kernel of a *-congruence in PCASL. We prove that if L is *-commutative PCASL in which $x \leq x^{**}$ for all $x \in L$ and K is filter of L, then the relation S_K on L defined by $(x, y) \in S_K$ if and only if $x \circ t = y \circ t$, for some $t \in K$ is the smallest *-congruence with co-kernel K. Also, we prove that the set $F^*(L)$ of all *-filters of PCASL L is a complete implicative lattice. Finally, we prove that if K is a *-filter of *-commutative PCASL then the *-congruence $S_K \vee \psi$ is the largest *-congruence with co-kernel K, where the relation ψ on L defined by $(x, y) \in \psi$ if and only if $x^{**} = y^{**}$ which is a *-congruence. First, we begin this section with the following.

If F is a filter of PCASL L such that $x \in F$, then it can be easily observed that $x^{**} \in F$. But, converse is not true. For, consider the following example.

4.1. Example: Let $L = \{0, a, b, c\}$. Now, define a binary operation $\circ$ on L as follows:

∘	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

It can be easily seen that L is an ASL and also L is PCASL under a unary operation * on L defined by $0^* = a, x^* = 0$, for all $x \neq 0 \in L$. Now, put $F = \{a, b\}$. Then clearly F is a filter of L. Now, consider $c^{**} = (c^*)^* = 0^* = a$. But, $c \notin F$. This motivate us to introduce *-filter in PCASL in the following.

4.2. Definition: Let L be a PCASL. Then a filter of L is said to be a *-filter if $x^{**} \in F$, then $x \in F$.

4.3.Example: Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ are two discrete ASLs. Let $L = A \times B = \{(0,0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. Define a binary operation $\circ$ on L as follows:

∘	(0,0)	$(0, b_1)$	$(0, b_2)$	$(a, 0)$	(a, b_1)	(a, b_2)

$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,b_1)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(a,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
(a,b_1)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_2)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)

Then clearly, $(L,\circ)$ is an ASL. Now, define a unary operation $*$ on L by $(0,0)^* = (a,b_1)$, $(0,b_1)^* = (0,b_2)^* = (a,0), (a,0)^* = (0,b_1)$ and $(a,b_1)^* = (a,b_2)^* = (0,0)$. Then clearly $*$ is a pseudo-complementation on L. Now, put $F = \{(a,b_1),(a,b_2)\}$. Then clearly, F is a *-filter of L.

Recall that if θ is a congruence relation on an ASL L, then the congruence class of the element 0 with respect to θ is called kernel of θ. Also, note that if θ is an arbitrary *-congruence on PCASL L and m_1, m_2 are any two unimaximal elements in L. Then $\theta_{m_1} = \theta_{m_2}$ is not known and investigation is going on. In the following we define co-kernel of a *-congruence on PCASL.

4.4. Definition: Let L be a PCASL and θ be a *-congruence on L. Then the congruence class of the element 0^* with respect to θ is called co-kernel of θ.

Turning our attention to filters, we prove that every filter of a PCASL is a co-kernel of a *-congruence. For, this first we need the following lemmas.

4.5. Lemma: Let L be a PCASL and K be a non-empty subset of L which is closed under $\circ$. Define a relation S_K on L by $(x,y) \in S_K$ if and only if $x \circ t = y \circ t$, for some $t \in K$ is a *-congruence on L.
Proof. Clearly, S_K is reflexive and symmetric. Let $(x,y),(y,z) \in S_K$. Then $x \circ t_1 = y \circ t_1$, $y \circ t_2 = z \circ t_2$, for some $t_1, t_2 \in K$. This implies $t_1 \circ t_2 \in K$. Now, consider $x \circ (t_1 \circ t_2) = \big((x \circ t_1) \circ t_2\big) = \big((y \circ t_1) \circ t_2\big) = \big((t_1 \circ y) \circ t_2\big) = \big(t_1 \circ (y \circ t_2)\big) = \big(t_1 \circ (z \circ t_2)\big) = \big((t_1 \circ z) \circ t_2\big) = \big((z \circ t_1) \circ t_2\big) = z \circ (t_1 \circ t_2)$. Therefore $(x,z) \in S_K$. Hence S_K is an equivalence relation on L. Let $(a,b),(c,d) \in S_K$. Then $a \circ t_1 = b \circ t_1$, $c \circ t_2 = d \circ t_2$, for some $t_1, t_2 \in K$. Now, consider $(a \circ c) \circ (t_1 \circ t_2) = ((a \circ c) \circ t_1) \circ t_2 = ((a \circ (c \circ t_1)) \circ t_2 = \big(a \circ (t_1 \circ c)\big) \circ t_2 = \big((a \circ t_1) \circ c\big) \circ t_2 = (a \circ t_1) \circ (c \circ t_2) = (b \circ t_1) \circ (d \circ t_2) = ((b \circ t_1) \circ d) \circ t_2 = (b \circ (t_1 \circ d)) \circ t_2 = (b \circ (d \circ t_1)) \circ t_2 = ((b \circ d) \circ t_1) \circ t_2 = (b \circ d) \circ (t_1 \circ t_2)$. Thereore $(a \circ c, b \circ d) \in S_K$. Hence S_K is an ASL congruence on L. Let $(x,0) \in S_K$. Then $x \circ t = 0 \circ t$, for some $t \in K$. This implies $x \circ t = 0$. It follows that $x^* \circ t = t$. Since 0^* is unimaximal, $x^* \circ t = 0^* \circ t$. Therefore $(x^*, 0^*) \in S_K$. Therefore S_K is a *-congruence on L.

4.6. Corollary: Let L be a PCASL and K be a filter of L. Then S_K is a *-congruence on L.

4.7. Lemma: Let L be a PCASL and K be a filter of L. Then for any two unimaximal elements $m_1, m_2 \in L$, the congruence classes $(S_K)_{m_1}, (S_K)_{m_2}$ of m_1, m_2 respectively are equal.
Proof. We have the congruence class of an element $x \in L$ with respect to the congruence relation S_K, that is $(S_K)_x = \{x \in L: (x,x) \in S_K\}$. Suppose m_1 and m_2 be two unimaximal elements in L. Let $y \in (S_K)_{m_1}$. Then $(y, m_1) \in S_K$. This implies $y \circ t = m_1 \circ t$, for some $t \in K$. It follows that $m_2 \circ (y \circ t) = m_2 \circ (m_1 \circ t)$. Since m_2 is unimaximal, $y \circ t = m_2 \circ t$. Therefore $(y, m_2) \in S_K$. Hence $y \in (S_K)_{m_2}$. Thus $(S_K)_{m_1} \subseteq (S_K)_{m_2}$. Similarly, we can prove that $(S_K)_{m_2} \subseteq (S_K)_{m_1}$. Therefore $(S_K)_{m_1} = (S_K)_{m_2}$

4.8. Lemma: Let L be a PCASL and K be a filter of L. Then for any unimaximal element m in L, $(S_K)_m$ is a filter.
Proof. We have $(S_K)_m = \{x \in L : (x,m) \in S_K\}$. Then clearly $x \in (S_K)_m$. Therefore $(S_K)_m$ is non-empty subset of L. Let $x, y \in (S_K)_m$. Then $(x,m),(y,m) \in S_K$. This implies $(x \circ y, m) \in S_K$. Therefore

$x \circ y \in (S_K)_m$. Again, let $x \in (S_K)_m$ and $a \in L$ such that $a \circ x = x$. This implies $(x, m) \in S_K$. It follows that $x \circ t = m \circ t$, for some $t \in K$. Therefore $a \circ x \circ t = a \circ m \circ t$. It follows that $x \circ t = m \circ a \circ t$. Since m is unimaximal,
$x \circ t = a \circ t$. Therefore $(x, a) \in S_K$ and $(x, m) \in S_K$. This implies $(a, m) \in S_K$. Hence $a \in (S_K)_m$. Thus $(S_K)_m$ is a filter.

4.9. Theorem: Let L be a PCASL and K be a filter of L. Then the co-kernel of S_K is K. Moreover $x \leq x^{**}$, for all $x \in L$ then S_K is the smallest *-congruence with co-kernel K.
Proof. Suppose $x \in (S_K)_{0^*}$. Then $(x, 0^*) \in S_K$. This implies $x \circ t = 0^* \circ t$, for some $t \in K$.
Since 0^* is unimaximal, $x \circ t = t$. It follows that $x \in K$, since K is filter. Therefore $(S_K)_{0^*} \subseteq K$. Conversely, suppose $x \in K$. Since $0^* \circ x = x = x \circ x$ and $x \in K$. It follows that $(0^*, x) \in S_K$. Therefore $x \in (S_K)_{0^*}$. Hence $K \subseteq (S_K)_{0^*}$. Thus $(S_K)_{0^*} = K$. Suppose $x \leq x^{**}$, for all $x \in L$. Let θ be a *-congruence on L with co-kernel K. i.e; $\theta_{0^*} = K$. Now, we shall prove that $S_K \subseteq \theta$. Let $(x, y) \in S_K$. Then $x \circ t = y \circ t$, for some $t \in K = \theta_{0^*}$. This implies $(t, 0^*) \in \theta$. Hence $(x \circ t, x \circ 0^*) \in \theta$. It follows that $(x \circ t, 0^* \circ x) \in \theta$, since $x \leq x^{**}, x \circ x^{**} = x^{**} \circ x$ and hence $0^* \circ (x \circ x^{**}) = 0^* \circ (x^{**} \circ x)$, we get $x \circ 0^* = 0^* \circ x$. Therefore $(x \circ t, x) \in \theta$. Similarly, we can prove that $(y \circ t, y) \in \theta$. It follows that $(x, y) \in \theta$. Thus S_K is the smallest *-congruence with co-kernel K.

We shall denote the set of *-filters of a PCASL L by $F^*(L)$. The following results, which show how the notation of a *-filters in a natural way, will allow us to investigate the structure of $F^*(L)$. First, we need the following.

4.10. Lemma: If F is a filter of a PCASL L, then $\alpha(F) = \{x \in L : x^* \in F\}$ is an ideal of L. Moreover, $\alpha(F)$ is a kernel ideal of L.
Proof. Suppose F is a filter of L. Since $0^* \in F, 0 \in \alpha(F)$. Therefore $\alpha(F)$ is non-empty subset of L. Let $a \in \alpha(F)$ and $t \in L$. Then $a^* \in F$ and $t \in L$. Now, we have $t \circ a \leq a$. It follows that $a^* \leq (t \circ a)^* = (a \circ t)^*$. This implies $(a \circ t)^* \in F$. Therefore $a \circ t \in \alpha(F)$. Hence $\alpha(F)$ is an ideal of L. Let $a, b \in \alpha(F)$. Then $a^*, b^* \in F$ and hence $a^* \circ b^* \in F$. Now, $((a^* \circ b^*)^*)^* = (a^* \circ b^*)^{**} = a^{***} \circ b^{***} = a^* \circ b^* \in F$. Therefore $((a^* \circ b^*)^* \in \alpha(F)$. Thus $\alpha(F)$ is a kernel ideal of L.

4.11. Lemma: If I is a kernel ideal of a PCASL L, then $\beta(I) = \{x \in L : x^* \in I\}$ is a *-filter of L.
Proof. Suppose I is a kernel ideal of L. Since $(0^*)^* = 0^{**} = 0 \in I$, $0^* \in \beta(I)$. Therefore $\beta(I)$ is non-empty subset of L. Let $x, y \in \beta(I)$. Then $x^*, y^* \in I$. Since I is a kernel ideal, $(x^{**} \circ y^{**})^* \in I$. This implies $(x \circ y)^{***} \in I$. It follows that $(x \circ y)^* \in I$. Therefore $x \circ y \in \beta(I)$. Let $x \in \beta(I)$ and $t \in L$ such that $t \circ x = x$. Since $t \circ x = x$, $t^* \circ t \circ x = t^* \circ x$. Therefore $t^* \circ x = 0$. Hence $x \circ t^* = 0$. It follows that $x^* \circ t^* = t^*$. Again, since $x^* \in I$, $x^* \circ t^* \in I$. Therefore $t^* \in I$. Hence $t \in \beta(I)$. Therefore $\beta(I)$ is a filter of L. Now, let $x^{**} \in \beta(I)$. Then $x^* = x^{***} \in I$. Hence $x \in \beta(I)$. Thus $\beta(I)$ is a *-filter.

It can be easily seen that the set $F(L)$ of al filters of a PCASL L form a complete lattice with respect to set inclusion. We can therefore define a mapping $\alpha : F(L) \rightarrow KI(L)$ by $F \mapsto \alpha(F)$ and $\beta : KI(L) \rightarrow F(L)$ by $I \mapsto \beta(I)$. In the following we prove that α is residuated mapping with residual map β.

4.12. Theorem: α is residuated with residual map β.
Proof. Clearly, $, \beta$ are isotone mappings. Also, we have $\beta(\alpha(F)) = \{x \in L : x^* \in \alpha(F)\} = \{x \in L : x^{**} \in F\}$. Let $x \in F$. Then we have $x^{**} \circ x = x$. It follows that $x^{**} \in F$. Therefore $x \in \beta(\alpha(F))$. Hence $F \subseteq \beta(\alpha(F))$. Let $I \in KI(L)$. Then consider $(\beta(I)) = \{x \in L : x^* \in \beta(I)\} = \{x \in L : x^{**} \in I\} = I$. Hence α is residuated and the residual of α is β.

4.13. Corollary: $\beta(\alpha(F)) = F$ if and only if F is *-filter.

Proof. Suppose $\beta(\alpha(F)) = F$ and suppose $x^{**} \in F$. Then $x \in \beta(\alpha(F)) = F$. Hence $x \in F$. Thus F is a *-filter. Conversely, suppose F is *-filter. We shall prove that $\beta(\alpha(F)) = F$. We have $F \subseteq \beta(\alpha(F))$. Let $x \in \beta(\alpha(F))$. Then $x^{**} \in F$. Therefore $x \in F$. Hence $\beta(\alpha(F)) \subseteq F$. Thus $\beta(\alpha(F)) = F$.

It follows from theorem 4.12, then $\beta \circ \alpha$ is a closure operator on the complete lattice $F(L)$. Using the corollary 4.13, of theorem 4.12, we can therefore assert:

4.14. Theorem: The set $F^*(L)$ of *-filters of L, ordered by set inclusion is a complete lattice in which the lattice operations are as follows: if $(F_\lambda)_{\lambda\in\Delta}$ is a family of *-filters of L, then $inf_{F^*(L)}\{F_\lambda: \lambda \in \Delta\} = \bigcap_{\lambda\in\Delta} F_\lambda$, $sup_{F^*(L)}\{F_\lambda: \lambda \in \Delta\} = (\beta \circ \alpha)\left(\bigcup_{\lambda\in\Delta} F_\lambda\right)$.

Proof. Proof follows by theorem 2.22.

Recall that $KI(L)$ is a complete implicative lattice. In the following we prove that $F^*(L)$ is complete implicative lattice.

4.15. Theorem: $F^*(L) \cong KI(L)$.

Proof. In view of lemma 4.12, β induces an isotone mapping $\widehat{\beta} : KI(L) \to F^*(L)$. Now, define $\widehat{\alpha} : F^*(L) \to KI(L)$ by $\widehat{\alpha}(F) = \alpha(F)$, for all $F \in F^*(L)$. Then by theorem 4.13, it follows that $\widehat{\beta} \circ \widehat{\alpha}$ and $\widehat{\alpha} \circ \widehat{\beta}$ are identity mappings. Thus $\widehat{\alpha}, \widehat{\beta}$ are mutually inverses. Thus $F^*(L)$ is isomorphic to $KI(L)$.

4.16. Corollary: $F(S(L)) \cong F^*(L) \cong KI(L) \cong I(S(L))$.

Proof. Proof follows by the theorem 3.8, apply theorem 4.15. to both $L, S(L)$.

4.17. Corollary: Suprema in $F^*(L)$ are given by $sup_{F^*(L)}\{F_\lambda: \lambda \in \Delta\} = \{x \in L : x^{**} \in \bigvee_{\lambda\in\Delta} F_\lambda\}$.

Proof. By theorem 4.15. we have $sup_{F^*(L)}\{(F_\lambda: \lambda \in \Delta\} = \widehat{\beta}\,(sup_{KI(L)}\{\widehat{\alpha}\,(F_\lambda): \lambda \in \Delta\})$. Now, we have $x \in \widehat{\alpha}\,(F_\lambda)$ if and only if $x^* \in F_\lambda$ and by theorem 2.22,

$$sup_{KI(L)}\{\widehat{\alpha}\,(F_\lambda): \lambda \in \Delta\} = \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists x_i \in \widehat{\alpha}\,(F_{\lambda_i})(x \le (\circ_{i=1}^{n}\ x_i{}^*)^*)\}.$$
$$= \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists x_i{}^* \in F_{\lambda_i})(x \le (\circ_{i=1}^{n}\ x_i{}^*)^*)\}.$$

Therefore $\widehat{\beta}\,(sup_{KI(L)}\{\widehat{\alpha}\,(F_\lambda): \lambda \in \Delta\})$

$$= \widehat{\beta}\,\{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists x_i{}^* \in F_{\lambda_i})(x \le (\circ_{i=1}^{n}\ x_i{}^*)^*)\}.$$
$$= \beta\,\{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists x_i{}^* \in F_{\lambda_i})(x \le (\circ_{i=1}^{n}\ x_i{}^*)^*)\}.$$
$$= \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists x_i{}^* \in F_{\lambda_i})(x^* \le (\circ_{i=1}^{n}\ x_i{}^*)^*)\}.$$

Thus we have

$$sup_{F^*(L)}\{(F_\lambda: \lambda \in \Delta\} = \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists y_i \in F_{\lambda_i})(x^* \le (\circ_{i=1}^{n} y_i\,)^*)\}.$$
$$= \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists y_i \in F_{\lambda_i})((\circ_{i=1}^{n} y_i\,)^{**} \le x^{**})\}$$
$$= \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists y_i \in F_{\lambda_i})(\circ_{i=1}^{n} y_i{}^{**}) \le x^{**})\}$$
$$= \{x \in L : (\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \triangle)(\exists y_i \in F_{\lambda_i})\ (\circ_{i=1}^{n} y_i) \le x^{**})\}.$$
$$= \{x \in L : x^{**} \in \bigvee_{\lambda\in\Delta} F_\lambda\}.$$

Finally, we prove the following.

4.8. Theorem: Let L be a *-commutative PCASL. If K is a *-filter of L, then the *-congruence $S_K \vee \psi$ has co-kernel K. In this case $S_K \vee \psi$ is the largest such *-congruence

Proof. Suppose K is a *-filter. Clearly, $S_K \vee \psi$ is a *-congruence on L. Now, we shall prove that co-kernel of $S_K \vee \psi$ is K. Let $x \in (S_K \vee \psi)_{0^*}$. Then $(x, 0^*) \in S_K \vee \psi$. This implies $(x^*, 0^{**}) \in S_K \vee \psi$. It follows that $x^{**} \circ t = 0^* \circ t$, for some $t \in K$. Hence $x^{**} \circ t = t$, since 0^* is unimaximal element. Therefore $x^{**} \in K$, since $t \in K$. Hence $x \in K$, since K is a *-filter. Thus $(S_K \vee \psi)_{0^*} \subseteq K$. Conversely, suppose $x \in K$. Since $0^* \in K$. It follows that $(x, 0^*) \in S_K$, since $x \circ x = x = 0^* \circ x$. Hence $(x, 0^*) \in S_K \vee \psi$. Therefore $x \in (S_K \vee \psi)_{0^*}$. Hence $K \subseteq (S_K \vee \psi)_{0^*}$ Thus $(S_K \vee \psi)_{0^*} = K$. Suppose θ is a *-congruence with co-kernel K. Now we shall prove that $\theta \subseteq S_K \vee \psi$. Let $(x, y) \in \theta$. Since $(x^*, x^*), (y^*, y^*) \in \theta$, $(x \circ x^*, y \circ x^*) \in \theta$. It follows that $(0, y \circ x^*) \in \theta$. Therefore $(y \circ x^*, 0) \in \theta$. Hence $((y \circ x^*)^*, 0^*) \in \theta$. It follows that $(x^* \circ y)^* \in \theta_{0^*}$ = K. Similarly, we can prove that $(x \circ y^*)^* \in \theta_{0^*} = K$. It follows that $(x^* \circ y)^* \circ (x \circ y^*)^* \in K$. Put $t = (x^* \circ y)^* \circ (x \circ y^*)^*$. Then $t \in K$. Consider $x^{**} \circ t = x^{**} \circ ((x^* \circ y)^* \circ (x \circ y^*)^*) = (x^{**} \circ (x^* \circ y)^*) \circ (x \circ y^*)^* = x^{**} \circ (x \circ y^*)^* = x^{**} \circ (x \circ y^*)^{***} = x^{**} \circ ((x \circ y^*)^*)^{**} = (x \circ (x \circ y^*)^*)^{**} = (x \circ (y^* \circ x)^*)^{**} = ((y^* \circ x)^* \circ x)^{**} = (y^{**} \circ x)^{**} = x^{**} \circ y^{**}$. Therefore $x^{**} \circ t = x^{**} \circ y^{**}$. Similarly, we can prove that $y^{**} \circ t = x^{**} \circ y^{**}$. It follows that $x^{**} \circ t = y^{**} \circ t$. Hence $(x^{**}, y^{**}) \in S_K$. But, we have (x, x^{**}), $(y^{**}, y) \in \psi$. Therefore there exists a finite sequence x, x^{**}, y^{**}, y such that $(x, x^{**}) \in \psi, (x^{**}, y^{**}) \in S_{K,}$ $(y^{**}, y) \in \psi$. Therefore $(x, y) \in S_K \vee \psi$. Hence $S_K \vee \psi$ is the largest *-congruence with co-kernel K

4.9. Corollary: If K is a *-filter then $(x, y) \in S_K \vee \psi \Leftrightarrow (x^* \circ y)^* \circ (x \circ y^*)^*$.

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