CONCEPTUAL FRAMEWORK ON UNITARY 2-REPRESENTATIONS OF FINITE GROUPS

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ABSTRACT

A ‘2-group’ is a category equipped with a multiplication satisfying laws like those of a group. Just as groups have representations on vector spaces, 2-groups have representations on ‘2-vector spaces’, which are categories analogous to vector spaces. Let Γ be a finite abelian group and let H be an infinite-dimensional separable complex Hilbert space. We prove that the set of realizable by action unitary representations of Γ on H is comeager in the space of all unitary representations of Γ on H.

Keywords: Unitary representation, Group action, vector spaces, Finite Groups

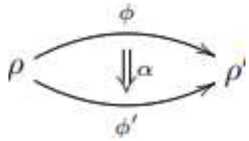
INTRODUCTION

The representation theory of finite groups is a subject going back to the late eighteen hundreds. The earliest pioneers in the subject were Frobenius, Schur and Burnside. Modern approaches tend to make heavy use of module theory and the Wedderburn theory of semisimple algebras. But the original approach, which nowadays can be thought of as via discrete Fourier analysis, is much more easily accessible and can be presented, for instance, in an undergraduate course. The aim of this textwork is to exposit the essential ingredients of the representation theory of finite groups over the complex numbers assuming only linear algebra and undergraduate group theory, and perhaps a minimal familiarity with ring theory

Unitary representations of groups play an important role in many subjects, including number theory, geometry, probability theory, partial differential equations, and quantum mechanics. This monograph focuses on dual spaces associated to a group, which are spaces of building blocks of general unitary representations.Special attention is paid to discrete groups for which the unitary dual, the most common dual space, has proven to be not useful in general and for which other duals spaces have to be considered, such as the primitive dual, the normal quasi-dual, or spaces of characters.

The work offers a detailed exposition of these alternative dual spaces and covers the basic facts about unitary representations and operator algebras needed for their study. The study of representations of skeletal 2-groups on measurable categories was begun by Crane and Yetter. The special case of the Poincar´e 2-group was studied by Crane and Sheppeard They noticed interesting connections to the orbit method in geometric quantization, and also to the theory of discrete subgroups of SO(3, 1), known as ‘Kleinian groups’. These observations suggest that Lie 2-group representations on measurable categories deserve a thorough and careful treatment. This, then, is the goal of the present text. We give geometric descriptions of:

- A representation ρ of a skeletal 2-group G on a measurable category H^X,
- An intertwine between such representations: $\rho \xrightarrow{\phi} \rho'$

- A 2-intertwiner between such intertwiners: 

We use the term ‘intertwiner’ as short for ‘intertwining operator’. This is a commonly used term for a morphism between group representations; here we use it to mean a morphism between 2-group representations. But in addition to intertwiners, we have something really new: 2-intertwiners between interwiners! This extra layer of structure arises from categorification

LITERATURE REVIEW

Marius Tarnauceanu (2019) in this paper, we study the finite groups whose Chermak-Delgado measure has exactly k values. We will focus especially on the case k = 2. These groups determine an interesting class of p-groups containing cyclic groups of prime order and extra special p-groups.

S. Kondrat’ev, N. V. Maslova& D. O. Revin (2018) A subgroup H of a group G is called pronormal if, for any element g of G, the subgroups H and Hg are conjugate in the subgroup they generate. Some problems in the theory of permutation groups and combinatory have been solved in terms of pronormality, and the characterization of pronormal subgroups in finite groups is a problem of importance for applications of group theory. A task of special interest is the study of pronormal subgroups in finite simple groups and direct products of such groups.

Yuemei Mao, Abid Mahboob &Wenbin Guo (2015) A subgroup H of a finite group G is said to be s-semipermutable in G if it is permutable with every Sylow p-subgroup of G with (p, |H|) = 1. We say that a subgroup H of a finite group G is S-semiembedded in G if there exists an s-permutable subgroup T of G such that TH is s-permutable in G and T ∩ H ⩽ [Math Processing Error], where [Math Processing Error] is an s-semipermutable subgroup of G contained in H. In this paper, we investigate the influence of S-semiembedded subgroups on the structure of finite groups.

B. Brewster, P. Hauck and E. Wilcox (2014) It is an open question in the study of Chermak-Delgado lattices precisely which finite groups G have the property that $\mathcal{CD}$(G) is a chain of length 0. In this note, we determine two classes of groups with this property. We prove that if G = AB is a finite group, where A and B are abelian subgroups of relatively prime orders with A normal in G, then the Chermak-Delgado lattice of G equals {ACB(A)}, a strengthening of earlier known results.

U. Acar, F. Koyuncu, and B. Tanay (2010) In order to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties, the notion of double-framed soft sets is introduced, and applications in BCK/BCI-algebras are discussed. The notions of double-framed soft algebras in BCK/BCI-algebras are introduced, and related properties are investigated. Characterizations of double-framed soft algebras are considered. Product and int-uni structure of double-framed soft algebras are discussed, and several examples are provided

What is Representation Theory?

Groups arise in nature as “sets of symmetries (of an object), which are closed under composition and under taking inverses”. For example, the symmetric group S_n is the group of all permutations (symmetries) of {1, n}; the alternating group an is the set of all symmetries preserving the parity of the number of ordered pairs (did you really remember that one?); the dihedral group D_{2n} is the group of symmetries of the regular n-gon in the plane. The orthogonal group O(3) is the group of distance-preserving transformations of Euclidean space which fix the origin. There is also the group of all distance-preserving transformations, which includes the translations along with O(3).1 The official definition is of course more abstract, a group is a set G with a binary operation ∗ which is associative, has a unit element e and for which inverses exist. Associatively allows a convenient abuse of notation, where we write gh for g ∗ h; we have ghk = (gh)k = g(hk) and parentheses are unnecessary. I will often write 1 for e, but this is dangerous on rare occasions, such as when studying the group Z under addition; in that case, e = 0. The abstract definition notwithstanding, the interesting situation involves group “acting” on a set. Formally, an action of a group G on a set X is an “action map” a: G × X → X which is compatible with the group law, in the sense that

$$a(h, a(g, x)) = a(hg, x) \text{ and } a(e, x) = x.$$

This justifies the abusive notation a(g, x) = g.x or even gx, for we have h(gx) = (hg)x. From this point of view, geometry asks, “Given a geometric object X, what is its group of symmetries?” Representation theory reverses the question to “Given a group G, what objects X does it act on?” and attempts to answer this question by classifying such X up to isomorphism. Before restricting to the linear case, our main concern, let us remember another way to describe an action of G on X. Every g ∈ G defines a map a(g) : X → X by x 7→ gx. This map is a bijection, with inverse map $a(g^{-1})$: indeed, $a(g^{-1}) \circ a(g)(x) = g^{-1} gx = ex = x$ from the properties of the action. Hence a(g) belongs to the set Perm(X) of bijective self-maps of X. This set forms a group under composition, and the properties of an action imply that

REPRESENTATIONS OF FINITE GROUPS

The representation theory of groups is a part of mathematics which examines how groups act on given structures. Here the focus is in particular on operations of groups on vector spaces. Nevertheless, groups acting on other groups or on sets are also considered. For more details, please refer to the section on permutation representations.With the exception of a few marked exceptions, only finite groups will be considered in this work. We will also restrict ourselves to vector spaces over fields of characteristic zero. Because the theory of algebraically closed fields of characteristic zero is complete, a theory valid for a special algebraically closed field of characteristic zero is also valid for every other algebraically closed field of characteristic zero. Thus, without loss of generality, we can study vector spaces C. Representation theory is used in many parts of mathematics, as well as in quantum chemistry and physics. Among other things it is used in algebra to examine the structure of groups. There are also applications in harmonic analysis and number theory. For example, representation theory is used in the modern approach to gain new results about automorphic forms.

Representation theory is the study of groups acting on vector spaces. It is the natural intersection of group theory and linear algebra. In math, representation theory is the building block for subjects like Fourier analysis, while also the underpinning for abstract areas of number theory likes the Langland’s program. It appears crucially in the study of Lie groups, algebraic groups, matrix groups over finite fields, combinatory, and algebraic geometry, just to name a few. In addition to great relevance in nearly all fields of mathematics, representation theory has many applications outside of mathematics. For example, it is used in chemistry to study the states of the hydrogen atom and in quantum mechanics to the simple harmonic oscillator. To start, I’ll try and describe some examples of representations, and highlight some strange coincidences. Much of our goal through the course will be to prove that these “coincidences” are actually theorems.

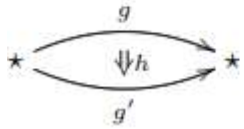
The representation theory of finite groups is a subject going back to the late eighteen hundreds. The earliest pioneers in the subject were Frobenius, Schur and Burnside. Modern approaches tend to make heavy use of module theory and the Wedderburn theory of semisimple algebras. But the original approach, which nowadays can be thought of as via discrete Fourier analysis, is much more easily accessible and can be presented, for instance, in an undergraduate course. The aim of this textbook is to exposit the essential ingredients of the representation theory of finite groups over the complex numbers assuming only linear algebra and undergraduate group theory, and perhaps a minimal familiarity with ring theory.

❖ 2-groups as 2-categories

We have said that a 2-group is a category equipped with product and inverse operations satisfying the usual group axioms. However, a more powerful approach is to think of a 2-group as a special sort of 2-category. To understand this, first note that a group G can be thought of as a category with a single object, morphisms labeled by elements of G, and composition defined by multiplication in G:

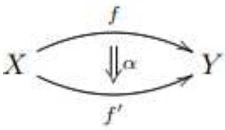
$$\star \xrightarrow{g_1} \star \xrightarrow{g_2} \star \quad = \quad \star \xrightarrow{g_2 g_1} \star$$

In fact, one can define a group to be a category with a single object and all morphisms invertible. The object * can be thought of as an object whose symmetry group is G. In a 2-group, we add an additional layer of structure to this picture, to capture the idea of symmetries between symmetries. So, in addition to having a single object * and its automorphisms, we have isomorphisms between automorphisms of *:

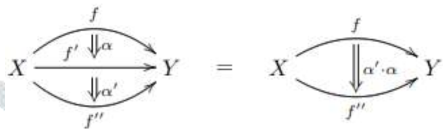


These ‘morphisms between morphisms’ are called 2-morphismsTo make this precise, we should recall that a 2-category consists of:

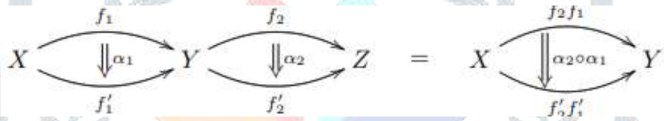
- objects: X, Y, Z
- morphisms: $X \xrightarrow{f} Y$
- 2-morphisms:



Morphisms can be composed as in a category, and 2-morphisms can be composed in two distinct ways: vertically:



And horizontally:

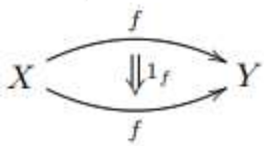


A few simple axioms must hold for this to be a 2-category

- Composition of morphisms must be associative, and every object X must have a morphism

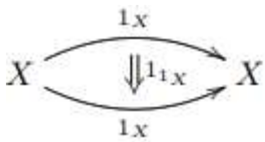
$$X \xrightarrow{1_x} X$$

- Serving as an identity for composition, just as in an ordinary category
- Vertical composition must be associative, and every morphism $X \xrightarrow{f} Y$ must have 2- morphism



Serving as an identity for vertical composition

Horizontal composition must be associative, and the 2-morphism

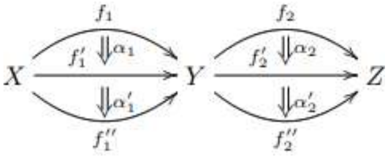


Must serve as an identity for horizontal composition

- Vertical composition and horizontal composition of 2-morphisms must satisfy the following exchange law:

$$(\alpha_2' \cdot \alpha_2) \circ (\alpha_1' \cdot \alpha_1) = (\alpha_2' \circ \alpha_1') \cdot (\alpha_2 \circ \alpha_1)$$

So that diagrams of the form



Define unambiguous 2-morphisms. For more details, see the references. We can now define a 2-group:

CONCLUSION

We have seen some properties of representations in general and of representations of finite groups in particular. We then looked at mathematical tools to find the irreducible representations of a group. Next, we classified the possible symmetry groups in crystals. All this mathematical groundwork can be a very powerful tool when we deal with systems with finite symmetries. To illustrate this point, we looked at an effect called crystal field splitting. We were able to make predictions about the energy spectrum of a crystal just from symmetry considerations. This shows what a powerful tool representation theory can be.

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