COMMON FIXED POINT THEOREMS ON BANACH SPACE USING CLR PROPERTY

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Abstract:-In this paper we discussed some fixed point results in cone Banach space and we introduce the concept of CLR (Common Limit Range) property with an example.
Keywords:-Cone Banach space, Common Fixed points, Compatible mapping, weakly compatible, CLR (Common Limit Range) property.

1. INTRODUCTION:-

Huang and Zhang [8] gave the notion of cone metric space, replacing the set of real numbers by ordered Banach Space and introduced some fixed point theorems for function satisfying contractive conditions in Banach Spaces. Sh. Rezapour and R. Hamalbarani [11] were generalized result of [8] by omitting the normality condition , which is milestone in developing fixed point theory in cone metric space. After that several articles on fixed point theorems in cone metric space were obtained by different mathematicians such as M. Abbas , G. Junck [10] , D. Ilic [2] etc. ErdalKarapinar[4], extended some of known results (1, 13) to cone Banach spaces which were defined and used in (3, 12)where the existence of fixed points for self-mappings on cone Banach spaces is investigated.In the generalization of non-compatible map Aamri[9] obtain E.A property, but it necessitates either the completeness of the whole space or any of the range spaces or map continuity. Sintunavarat and Kumam [14] recently proposed a novel idea of the CLR property (common limit range property) that does not impose such constraints. The importance of the CLR property ensures that range subspace closeness is not required.WutipholSintunavarat and PoomKumam (2011)[14] prove some common fixed point theorems under weakly compatible mappings using new property that is CLR (Common Limit Range) property. Recently many mathematicians use this property (Common Limit Range Property) in different spaces to prove fixed point theorem. In 2013 Jitendra Kumar[6] put his efforts to proved common fixed point theorem in Metric Space by Property (E.A.) as well as CLR property and given an example and prove common fixed point theorem from both methods and H.K. Pathak [5] prove their main theorem using (CLR) property for two pairs of self-mappings in complex-valued metric space under weak compatibility. In 2016 KPR Rao[7] proved common coupled fixed point theorem in partial metric space by CLR property.

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A subset $P \subseteq E$ is said to be a cone if and only if

(i). P is closed, nonempty and $P \neq \{0\}$
(ii). $a, b \ R, a, b \geq 0,\ u, v \in P$ implies $au,\ bv \in P$
(iii). $P \cap (-P) = \{0\}$

For a given cone P subset of E, we define a partial ordering $\leq$ with respect to P by $u \leq v$ if and only if $v - u \in P$. We shall write $u < v$ to indicate that $u \leq v$ but $u \neq v$ while $u << v$ will stand for $v - u \in int\ P$ where $int\ P$ denotes interior of **P** and is assumed to be nonempty.

Definition 2.2. Let $\mathbb{M}$ be a nonempty set. Suppose that the mapping $d : \mathbb{M} \times \mathbb{M} \rightarrow E$ satisfies

(i). $0 \leq \mathrm{d(u, v)}$for every $\mathrm{u, v} \in \mathbb{M}, \mathrm{d(u, v)} = 0$ if and only if $\mathrm{u = v}$.
(ii). $\mathrm{d(u, v)} = \mathrm{d(v, u)}$for every $\mathrm{u, v} \in \mathbb{M}$.
(iii). $d(u, v) \leq d(u, w) + d(w, v)$for every $u, v, w \in \mathbb{M}$.

Then d is a cone metric on $\mathbb{M}$and $(\mathbb{M}, d)$ is a cone metric space.

Definition 2.3. Two self maps A and B on a cone normed space $(\mathbb{M}, \|\cdot\|)$ are said to be weak-compatible if they commute at their coincidence points, i.e. $Au = Bu$ implies

$$ABu = BAu.$$

Definition 2.4. Let $\mathbb{M}$ be a vector space over R. Suppose the mapping $\|\cdot\| : \mathbb{M} \to E$ satisfies

(i). $\|u\| \geq 0$ for all $u \in \mathbb{M}$
(ii). $\|u\| = 0$ if and only if $u = 0$
(iii). $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathbb{M}$
(iv). $\|ku\| = |k| \|u\|$ for all $k \in R$.

Then $\|\cdot\|$ is called a norm on $\mathbb{M}$, and $(\mathbb{M}, \|\cdot\|)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with metric defined by $d(u, v) = \|u - v\|$

Definition 2.5. Let $(\mathbb{M}, \|\cdot\|)$ be a cone normed space, $u \in \mathbb{M}$ and $\{u_p\}$ a sequence in $\mathbb{M}$. Then

(i). $\{u_p\}$converges to uif for every $c \in E$ with $0 \ll c$ there is a natural number N such that

$$\|u_p - u\| \leq c \text{ for all } n \geq N$$

we shall denote it by $\lim_{p\to\infty} u_p = u$ or $u_p \to u$.

(ii). $\{u_p\}$ is a Cauchy sequence, if for every $c \in E$ with $0 \ll c$ there is a natural number N such that

$$\|u_p - u_m\| \leq c \text{ for all } p, m \geq N$$

(iii). $(\mathbb{M}, \|\cdot\|)$is a complete cone normed space if every Cauchy sequence is convergent. A complete cone normed space is called a Cone Banach space.

Definition 2.6. Let F and G be self mappings on a cone normed space $(\mathbb{M}, \|\cdot\|)$, they are said to be compatible if $\lim_{p\to\infty} \|FG(u_p) - GF(u_p)\| = 0$ for every sequence $\{u_p\}$ in $\mathbb{M}$ with $\lim_{p\to\infty} F(u_p) = \lim_{p\to\infty} G(u_p) = v$ for some point v in $\mathbb{M}$.

Proposition 2.7. Let $(\mathbb{M}, \|\cdot\|)$ be a cone normed space. P be a normal cone with constant K. Let $\{u_p\}$ be a sequence in $\mathbb{M}$. Then

(i). $\{u_p\}$ converges to x if and only if $\|u_p - u\| \to 0$ as $p \to \infty$.
(ii). $\{u_p\}$ is a Cauchy sequence if and only $\|u_p - u_m\| \to 0$ as $p, m \to \infty$.
(iii). If the $\{u_p\}$ converges to u and $\{y_n\}$converges to vthen $\|u_p - v_p\| \to \|u - v\|$

Proposition 2.8. Let $F$and G be compatible mappings on a cone normed space $(\mathbb{M}, \|\cdot\|)$ such that $\lim_{p\to\infty} F(u_p) = \lim_{p\to\infty} G(u_p)$ for some point v in $\mathbb{M}$and for every sequence $\{u_p\}$ in $\mathbb{M}$. Then $\lim_{p\to\infty} F(u_p) = F(v)$. if F is continuous.

3. Main Result:

Definition 3.1: Suppose that $(\mathbb{M}, \|\cdot\|)$ is cone Banach space and $F, G: \mathbb{M} \to \mathbb{M}$, Two mapping F and G are said to satisfy the common limit in the range of G property if

$$\lim_{p\to\infty} Fu_p = \lim_{p\to\infty} Gu_p = Gu \text{ for some } u \in \mathbb{M}$$

Example 3.2: Let $\mathbb{M} = [0, \infty)$ be the usual metric space. Define $F, G: \mathbb{M} \to \mathbb{M}$ by $F(u) = 2u + 1$ and $G(u) = 3u$ for all $u \in \mathbb{M}$. We consider the sequence $\{u_p\} = \left[1 + \frac{1}{p}\right]$

Since

$$\lim_{p\to\infty} Fu_p = \lim_{p\to\infty} Gu_p = 3 = 3 \cdot 1$$

Therefore F and G satisfy the (CLR_G) property.

Theorem 3.3: Let F and G be mappings on cone Banach Space $(\mathbb{M}, ||\cdot||)$ into itself with $||u|| = d(u, 0)$ satisfying the conditions

$$\|F(u) - F(v)\| \leq a\|F(u) - G(u)\| + b \begin{bmatrix} max \begin{Bmatrix} \|G(u) - F(u)\|, \\ \|G\ (u) - G(v)\| \end{Bmatrix} \\ +\|G(u) - F(v)\| \end{bmatrix} \quad \dots (1)$$

Where a and b are non negative and $a + 2b < 1$ and $u, v \in \mathbb{M}$

(i). $\mathrm{F}(\mathbb{M}) \subseteq \mathrm{G}(\mathbb{M})$
(ii). The pair (G, F) weakly compatible.
(iii). Pair (F, G) satisfies $(\mathrm{CLR_F})$ or $(\mathrm{CLR_G})$ property

Then F and G have a unique common fixed point.

Proof: First of all, assume (F, G) satisfy the (CLR_G) property then by the definition of (CLR_G) property there exist a sequence $\{u_p\}$ in $\mathbb{M}$ such that

$$\lim_{p\to\infty} F\{u_p\} = \lim_{p\to\infty} G\{u_p\} = Gu$$

Now we show that $F(u) = G(u)$ for some $u \in \mathbb{M}$, on the contradiction we put $u = u_p$ and $v = u$ in (1)

$$\|F(u_p) - F(u)\| \leq a\|F(u_p) - G(u_p)\| + b \begin{bmatrix} max \begin{Bmatrix} \|G(u_p) - F(u_p)\|, \\ \|G\ (u_p) - G(u)\| \end{Bmatrix} \\ +\|G(u_p) - F(u)\| \end{bmatrix}$$

Taking limit $p \to \infty$

$$\|G(u) - F(u)\| \leq a.0 + \ b \begin{bmatrix} max\ \{\|G\ (u) - G(u)\|, \|G\ (u) - G(u)\|\} \\ +\|G(u) - F(u)\| \end{bmatrix}$$

$$(1 - b)\|G(u) - F(u)\| \leq 0$$

But $(1 - b) \neq 0$ then $G(u) = F(u)$

Next, Let $w = F(u) = G(u)$, F and G are weakly compatible $FGu = GFu$

$$Fw = FGu = GFu = Gw$$

We claim that $Fw = w$ on the contradiction we put $u = w \ and \ v = u$ in (1)

$$\Rightarrow \|F(w) - F(u)\| \leq a\|F(w) - G(w)\| + \ \ b \begin{bmatrix} max \begin{Bmatrix} \|G(w) - F(w)\|, \\ \|G\ (w) - G(u)\| \end{Bmatrix} \\ +\|G(w) - F(u)\| \end{bmatrix}$$

$$\Rightarrow \|F(w) - w\| \leq a.0 + b \begin{bmatrix} max\{\|0\|, \|F\ (w) - w\|\} \\ +\|F(w) - w\| \end{bmatrix}$$

$\Rightarrow (1 - 2b)\|F(w) - w\| \leq 0$

But $(1 - 2b) \neq 0$ then $F(w) = w$

Hence $F(w) = w$ i.e. $w = F(w) = G(w)$,

Therefore w is a common fixed point of F and G.

<u>For uniqueness of a common fixed point</u>

Let F and G have another fixed point that is w, for this we put $u = t'$ & $v = w$ in (1)

$$\|F(t') - F(w)\| \ \ \leq a\|F(t') - G(t')\| + b \begin{bmatrix} max \begin{Bmatrix} \|G(t') - F(t')\|, \\ \|G\ (t') - G(w)\| \end{Bmatrix} \\ +\|G(t') - F(w)\| \end{bmatrix}$$

$\Rightarrow \quad \|t' - w\| \leq a.0 + b\,[max\{0, \|t' - w\|\} + \|t' - w\|]$

$\Rightarrow \quad (1 - 2b)\|t' - w\| \leq 0$

But $(1 - 2b) \neq 0$ then $t' = w$

Hence w is the unique common fixed point of mappings F and G.

Theorem 3.4: Let F, G, H, and L be mappings on cone Banach space $(\mathbb{M}, ||.||)$ into itself with $||u|| = d(u, 0)$ satisfying the conditions

$$\|H(u) - L(v)\| \leq a \max\begin{Bmatrix}\frac{1}{2}\|F(u) - L(v)\|, \\ \|F(u) - H(u)\|\end{Bmatrix} + b\|H(u) - G(v)\| + c\|F(u) - H(u)\| \;\ldots(1)$$

For all $u, v \in \mathbb{M}$, $a, b, c \geq 0$ and $a + 2b + c < 1$

(i). $\mathrm{H}(\mathbb{M}) \subseteq \mathrm{G}(\mathbb{M})$ and $\mathrm{L}(\mathbb{M}) \subseteq \mathrm{F}(\mathbb{M})$

(ii). Pairs (H, F) and (L, G) are weakly compatible.

(iii). Pair (L, G)or (H, F) satisfy $(\mathrm{CLR_L})$ or $(\mathrm{CLR_H})$property respectively

Then F , G , H and L have a unique common fixed point.

Proof: First we assume that the pair (L, G) satisfy (CLR_L) property then by the definition of (CLR_L) there exist a sequence $\{u_p\}$ in $\mathbb{M}$ such that

$$\lim_{p\to\infty} Lu_p = \lim_{p\to\infty} Gu_p = Lu \quad \text{for some } u \in \mathbb{M}$$

Further since $L(\mathbb{M}) \subseteq F(\mathbb{M})$ we have $Lu = Fw$ for some $w \in \mathbb{M}$.
We claim that
$Hw = Fw = t$ (say). If not then put $u = w$ and $v = u_p$ in (2)

$$\|H(w) - L(u_p)\| \leq a \max\begin{Bmatrix}\frac{1}{2}\|F(w) - L(u_p)\|, \\ \|F(w) - H(w)\|\end{Bmatrix} + b\|H(w) - G(u_p)\| + c\|F(w) - H(w)\|$$

Letting $p \to \infty$ and using above condition we get

$$\|H(w) - L(u)\| \leq a\,max\begin{Bmatrix}\frac{1}{2}\|F(w) - L(u)\|, \\ \|F(w) - H(w)\|\end{Bmatrix} + b\|H(w) - L(u)\| + c\|F(w) - H(w)\|$$

$$\|H(w) - L(u)\| \leq a\,max\left\{\frac{1}{2}.0, \|L(u) - H(w)\|\right\} + b\|H(w) - L(u)\| + c\|L(u) - H(w)\|$$

$$(1 - a - b - c)\|H(w) - L(u)\| \leq 0$$

Hence $Hw = Lu$ implies that $Fw = Hw = Lu = t$.
Hence w is coincidence point of $H$and F.
Since the pair (H, F) is weak compatible
$\Rightarrow HFw = FHw = Ht = Ft$
Further since $H(\mathbb{M}) \subseteq G(\mathbb{M})$ there exist some $z \in \mathbb{M}$ such that

$$H(w) = G(z)$$

We claim that $L(z) = t$ on the contradiction we put
$u = w$ and $v = z$ in (2)

$$\|H(w) - L(z)\| \leq a \max\begin{Bmatrix}\frac{1}{2}\|F(w) - L(z)\|, \\ \|F(w) - H(w)\|\end{Bmatrix} + b\|H(w) - G(z)\| + c\|F(w) - H(w)\|$$

Letting $p \to \infty$ and using above condition we get

$$\|t - L(z)\| \leq a \max \left\{ \begin{matrix} \frac{1}{2}\|t - L(z)\|, \\ \|H(w) - H(w)\| \end{matrix} \right\} + b\|G(z) - G(z)\| + c\|H(w) - H(w)\|$$

$$\|t - L(z)\| \leq a \max \left\{ \frac{1}{2}\|t - L(z)\|, 0 \right\} + b\|t - L(z)\| + c.0$$

$$\left(1 - \frac{a}{2}\right)\|t - L(z)\| \leq 0$$

Thus $Lz = t$ hence $Fw = Hw = Lz = Gz = t$ it shows that z is coincidence point of L and G.
Also the weak compatibility of (L, G) implies that

$$LGz = GLz = Lt = Gt$$

We claim that t is common fixed point of F, G, H and L, on the contradiction let us put $u = w$ and $v = t$ in (2)

$$\|H(w) - L(t)\| \leq a \max \left\{ \begin{matrix} \frac{1}{2}\|F(w) - L(t)\|, \\ \|F(w) - H(w)\| \end{matrix} \right\} + b\|H(w) - G(t)\| + c\|F(w) - H(w)\|$$

Letting $p \to \infty$ and using above condition we get

$$\|t - L(t)\| \leq a \max \left\{ \begin{matrix} \frac{1}{2}\|t - L(t)\|, \\ \|H(w) - H(w)\| \end{matrix} \right\} + b\|t - L(t)\| + c\|H(w) - H(w)\|$$

$$\|t - L(t)\| \leq a \max \left\{ \frac{1}{2}\|t - L(t)\|, 0 \right\} + b\|t - L(t)\| + c.0$$

$$(1 - \frac{a}{2} - b)\|t - L(t)\| \leq 0$$

Thus $Lt = t$ hence, $Ft = Ht = Lt = Gt = t$.
It shows that t is common fixed point of F, G, H and L. In the simple way, calculate uniqueness.
Similarly, the argument that the pair (H, F) satisfy property (CLR_H) will also give athe unique common fixed point of F, G, H and L. As a result, we arrive at the same conclusion in both cases: the existence and uniqueness of the common fixed point of $F, G, H,$ and L.

References

[1]. Ciric, Fixed Point Theory Contraction Mapping principle, Faculty of Mechanical Engineering Press, Beograd, Serbia, 2003.

[2]. D. Ilic and V. Rakoevic, Common fixed points for maps on cone metric space, J.Math. Anal. Appl. 341 (2) (2008) 876.

[3]. D. Turkoglu, M. Abuloha, and T. Abdeljawad, "Some theorems and examples of cone metric spaces," to appear in Journal of Computational Analysis and Applications.

[4]. ErdalKarapinar, Fixed point Theorems in Cone Banach Space, Hindawi Publishing Corporation, Fixed point Theory and Applications, Article ID 609281, (2009) 9.

[5]. H. K. Pathak, "Common Fixed Point Theorems Using Property (E.A) In Complex-Valued Metric Spaces" Thai Journal of Mathematics Volume 11 Number 2 : 347–355 (2013).

[6]. Jitender Kumar, "Common Fixed Point Theorems Of Weakly Compatible Maps Satisfying (E.A.) And (Clr) Property" International Journal of Pure and Applied Mathematics Volume 88 No. 3, 363-376 (2013).

[7]. K. P. R. Rao, "Common Fixed Point Theorems For Mappings Under (Clrs)-Property In Partial Metric Spaces" Demonstratio Mathematica , Vol. 49 No 3 (2016).

[8]. L G. Huang, X. Zhang, Cone metric space and fixed point theorems, Math. Anal. Appl. 332 (2) (2007) 1468. Anal. Appl. 341 (2) (2008) 876.

[9]. M. Aamri and D. El Moutawakil, “Some new common fixed point theorems under strict contractive conditions,” Journal of Mathematical Analysis and Applications, vol. 270, no. 1, pp. 181– 188, 2002.

[10]. M. Abbas, G. Jungck, Common fixed point results for non-commuting mappings without continuity in cone metric space, J. Math. Anal. Appl. 341 (2008)416.

[11]. Sh. Rezapour and R. Hamlbarani, “Some notes on the paper: “Cone metric spaces and fixed point theorems of contractive mappings”,” Journal of Mathematical Analysis and Applications, vol. 345, no. 2, pp. 719–724, 2008.

[12]. T. Abdeljawad, “Completion of cone metric spaces,” to appear in Hacettepe Journal of Mathematics and Statistics.

[13]. T. Suzuki, “Fixed point theorems and convergence theorems for some generalized non-expansive.

[14]. W. Sintunavarat and P.Kumam, “Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces,” Journal of Applied Mathematics, vol. 2011, Article ID 637958, 14 pa.