Köthe- Toeplitz Duals of Generalized Difference Sequence Spaces

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Abstract: In this paper we introduce the difference sequence spaces $c_0(\Delta_v)$, $c(\Delta_v)$, $l_\infty(\Delta_v)$, and $bv(\Delta_v)$. The properties of difference sequence spaces i.e. AK-Property, BK-Space, normality, Schauder basis and Köthe - Toeplitz duals are also the part of our studies. We extentded the work done by Kizmaz on the difference sequence spaces $c_0(\Delta), c(\Delta)$ and $l_\infty(\Delta)$ and V. K. Bhardwaj and Sandeep Gupta on Cesaro summable difference sequence spaces $C_1(\Delta)$.

Keywords- Difference Sequence Spaces, BK-space, Schauder basis, $\alpha-$, $\beta-$ *and* $\gamma-$ *duals* and Köthe - Toeplitz duals.

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Definitions

Let ω denote the linear space of all complex sequence spaces over $\mathbb{C}$ (the field of complex numbers). A vector subspace of ω is called a sequence subspace. l_∞, c, c_0 denote the space of all bounded, convergent and null sequence spaces $x = (x_k)$ with complex terms respectively. By cs we denote the space of all convergent series and bv denotes the space of all sequences of bounded variation.

Throughout this paper, unless otherwise specified we denote $\sum_k$ for $\sum_{k=1}^{\infty}$ and $\lim_n$ for $\lim_{n\to\infty}$.

A complete metric linear space is called a Fréchet space. Let X be a linear subspace of ω st. X is a Fréchet space with continuous coordinate projections. Then we say that X is FK-space. If the metric of a FK-space is given by a complete norm, then we say that X is BK-space.

We say that an FK-space X has AK or has the AK-property, if (e_k) the sequence of unit vectors, is a Schauder basis for X.

A sequence space X is called
(i) normal if $y = (y_k) \in X$ whenever $|y_k| < |x_k|$, $k \geq 1$ for some $x = (x_k) \in X$η
(ii) monotone if it contains the canonical preimages of its step spaces.
(iii) a sequence algebra if $xy = (x_k y_k) \in X$ whenever $x = (x_k), y = (y_k) \in X$.
(iv) symmetric if $(x_k) \in X \Rightarrow \left(x_{\pi(k)}\right)$ where π is permutation on $\mathbb{N}$.

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The idea of dual sequence spaces was introduced by Köthe and Toeplitz whose main result concerned $\alpha - duals$, the $\alpha - dual$ of $X \subset \omega$ being defined as-

$$X^{\alpha} = \left\{ a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty \text{ for all } x = (x_k) \in X \right\}$$

They also introduced another kind of duals namely the $\beta - dual$ by Chilling-worth[12] ,

$$X^{\beta} = \left\{ a = (a_k) \in \omega : \sum_k a_k\, x_k \text{ converges for all } x = (x_k) \in X \right\}$$

A still more general notion of dual was introduced by Garling in Kamthan and Gupta[7] as

$$X^{\gamma} = \left\{ a = (a_k) \in \omega : \sup_k \left| \sum_{i=1}^{k} a_i x_i \right| < \infty \text{ for all } x = (x_k) \in X \right\}$$

A more general notation of dual was introduced by Chandra and Tripathy[16] ,

$$X^{\eta} = \{ a = (a_k) \in \omega : (a_k x_k) \in I_r \text{ for all } x = (x_k) \in X \}$$

Obviously, $\phi \subset X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$, where ϕ is the well-known sequence spaces of finitely non-zero scalar sequences. Also if $X \subset Y$ then $Y^t = X^t$ for $t = \alpha, \beta, \gamma$. For any sequence space X, we denote $\left(X^{\delta}\right)^t$ by $X^{\delta t}$ where $\delta, t = \alpha,\ \beta\ or\ \gamma$. It is clear that $X \subset X^{tt}$ where $t = \alpha,\ \beta\ or\ \gamma$.
For a sequence space X, if $X = X^{\alpha\alpha}$ then X is called a Köthe space or a perfect space. The notion of difference sequence spaces was introduced by Kizmaz in 1981 as follows:

$$X(\Delta) = \{ x = (x_k) \in \omega : (\Delta x_k) \in X \}$$

for $X = l_{\infty},\ c,\ c_0$; where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$.

Introduction

The difference sequence spaces have been studied by Cooke[10] , Maddox[2] , Colak[4] , Orhan[5] and Bektas[12] . Soon after the introduction of the notion of difference sequence space, Orhan[6] ,[7] in 1983, applied the same technique of taking differences to the Cesaro spaces ces_p. $1 \le p \le \infty$ and ces_{∞} of Shiue[4] to introduce the Cesaro difference sequence spaces C_p, $1 \le p \le \infty$ and C_{∞}. Malkowsky[1] studied the matrix transformations in difference sequence spaces.
Since the initiation of the study of difference sequence spaces by kizmaz[3] , a large number of literature has grown. The work of this paper is an extention of the work done by Kizmaz [3] on the difference sequence spaces $c_0(\Delta), c(\Delta)$, and $l_{\infty}(\Delta)$ and V. K. Bhardwaj and Sandeep Gupta[13] on Cesaro summable difference sequence spaces $c_1(\Delta)$.
Let $v = (v_k) = (1,1,1,\dots)$ be any fixed sequence of non-zero complex numbers. Now we define

$$X(\Delta_v) = \{ x = (x_k) \in \omega : (\Delta_v x_k) \in X \}$$

for $X = l_{\infty},\ c,\ c_0,\ C_1,\ bv$; where $\Delta_v x_k = v_k x_k - v_{k+1} x_{k+1}$, for all $k \in \mathbb{N}$.
i.e.

$$l_{\infty}(\Delta_v) = \{ x = (x_k) \in \omega : (\Delta_v x_k) \in l_{\infty} \};$$

$$c(\Delta_v) = \{ x = (x_k) \in \omega : (\Delta_v x_k) \in c \};$$

$$c_0(\Delta_v) = \{ x = (x_k) \in \omega : (\Delta_v x_k) \in c_0 \};$$

$$C_1(\Delta_v) = \{x = (x_k) \in \omega : (\Delta_v x_k) \in C_1\};$$

$$bv(\Delta_v) = \{x = (x_k) \in \omega : (\Delta_v x_k) \in bv\}.$$

The overall picture regarding inclusion among the already existing spaces l_∞, c, c_0, C_1 and the newly introduced space $l_\infty(\Delta_v)$, $c(\Delta_v)$, $c_0(\Delta_v)$, $C_1(\Delta_v)$ and $bv(\Delta_v)$ as shown below-

$$\begin{array}{ccccccccc} & & & & C_1 & \subset & C_1(\Delta_v) \\ & & & & \cup & & \cup \\ l_1 & \subset & bv(or\ c_0) & \subset & c & \subset & l_\infty \\ \cap & & \cap & & \cap & & \cap \\ l_1(\Delta_v) & \subset & bv(\Delta_v)\big(or\ c_0(\Delta_v)\big) & \subset & c(\Delta_v) & \subset & l_\infty(\Delta_v) \\ & & & & & & \cap \\ & & & & & & C_1(\Delta_v) \end{array}$$

In section 3, different inclusion relations that are strict are disscussed. Also different topological properties of $C_1(\Delta_v)$ are also discussed. Section 4, is devoted to study of Köthe-Toeplitz and γ duals of these spaces.

The Inclusion Relations

Theorem 3.1 $l_\infty \subset C_1(\Delta_v)$, *the inclusion being strict.*
Proof Let $x = (x_k) \in l_\infty$. Then there exists $M > 0$ such that $|x_k - x_{k+1}| < M$ for all $k \geq 1$, and so $\frac{1}{k}\sum_{i=1}^{k} \Delta\, v_i x_i \to 0$ as $k \to \infty$. For strict inclusion, observe that $(k) \in C_1(\Delta_v)$ but $k \notin l_\infty$.

Theorem 3.2 $C_1 \subset C_1(\Delta_v)$, *the inclusion being strict.*
Proof For $x = (x_k) \in C_1$, we have $\lim_k \frac{1}{k} x_k = 0$, and so $\frac{1}{k}\sum_{i=1}^{k} \Delta\, v_i x_i \to 0$ as $k \to \infty$. Inclusion is strict in view of the example in theorem 3.1.

Theorem 3.3 $bv \subset bv(\Delta_v)$, *the inclusion being strict.*
Proof Let $x = (x_k) \in bv$. Then $(\Delta v_k x_k) \in l_1 \subset bv$. For strict inclusion, consider the sequence $x = (x_0, x_1, x_2, \dots)$ where

$$x_k = \begin{cases} 0, & \text{for } k = 0 \\ k(-1) - (k-1)\frac{1}{2} - (k-2)\frac{1}{2^2} \dots - \frac{1}{2^{k-1}}, & \text{for } k \geq 1. \end{cases}$$

Similarly, $c_0 \subset c_0(\Delta_v)$, $c \subset c(\Delta_v)$, $l_\infty \subset l_\infty(\Delta_v)$ these inclusions are also strict.

Theorem 3.4 $c(\Delta_v) \subset C_1(\Delta_v)$, *the inclusion being strict.*
Proof Inclusion is obvious since $c \subset C_1$. To see that the inclusion is strict, consider the sequence $x = (x_k) = (1,2,1,2,1,2,\dots)$.

Theorem 3.5 $l_1(\Delta_v) \subset bv(\Delta_v) \subset c(\Delta_v)$, *the inclusion being strict.*
Proof The result follows from the fact that $l_1 \subset bv \subset c$. For strict inclusion $l_1(\Delta_v) \subset bv(\Delta_v)$, observe

that $(k) \in bv(\Delta_v)$ but $(k) \in l_1(\Delta_v)$. Inclusion $bv(\Delta_v) \subset c(\Delta_v)$ is strict as $(y_k) = \left(0, -1, -1 + \frac{1}{2}, -1 + \frac{1}{2} - \frac{1}{3}, \dots\right) \in c(\Delta_v)$ but does not belong to $bv(\Delta_v)$.

Theorem 3.6 *$C_1(\Delta_v)$ and $bv(\Delta_v)$ are BK-space with the norm* $\|x\|_{\Delta_v} = |x_1| + sup_k \frac{1}{k} |\sum_{i=1}^{k} \Delta\, v_i x_i|$, $\quad x = (x_k) \in C_1(\Delta_v)$ *and* $\|x\|_{\Delta_v} = |x_1| + |x_2| + \sum_k |\Delta v_k x_k - \Delta v_{k+1} x_{k+1}|$, $\quad x = (x_k) \in bv(\Delta_v)$, *respectively.*

Proof Since $C_1(\Delta_v)$ and $bv(\Delta_v)$ are Banach spaces with continuous coordinates, that is, $||x^s - x||_{\Delta\vartheta} \to 0$ implies $|x_k^s - x_k| \to 0$, for each $k \in \mathbb{N}$, as $s \to \infty$, they are also BK-spaces.

Theorem 3.7 $bv(\Delta_v)$ does not have the AK-property.

Proof Let $x = (x_k) = (1,2,3,\dots) \in bv(\Delta_v)$. Consider the n^{th} section of the sequence (x_k) as $x^{[n]} = (1,2,3,\dots,n,0,0,0,\dots)$.

Then

$$\begin{aligned} ||x - x^{[n]}||_{b\vartheta} &= ||(0,0,0,\dots,n+1,n+2,\dots)||_{b\vartheta} \\ &= |0| + |0| + \dots + |n+1| + |n+2| + \dots \end{aligned}$$

which does not tend to 0 as $n \to \infty$

Theorem 3.8 $bv(\Delta_v)$ is not monotone.

Proof Take $(x_k) = (1,1,1,\dots) \in bv(\Delta_v)$ and $y = y_k$ as

$$y_k = \begin{cases} x_k, & \text{if k is odd} \\ 0, & \text{if k is even} \end{cases}$$

i.e., $(y_k) = (1,0,1,0,1,0,\dots)$. Then $(\Delta_v y_k) = (1,-1,1,-1,\dots)$ and so $(y_k) \notin bv(\Delta_v)$.

Theorem 3.9 $bv(\Delta_v)$ *is neither symmetric nor a sequence algebra.*

Proof Let $(x_k) = (1,2,3,4,\dots) \in bv(\Delta_v)$ and $(y_k) = (2,1,4,3,6,\dots)$ e a rearrangement of the terms of the sequence (x_k). Here $(\Delta_v y_k) = (1,-3,1,-3,\dots) \in bv$ and so $(y_k) \in bv(\Delta_v)$. This shows that $bv(\Delta_v)$ is not a symmetric space.

Lemma 3.10 Let X be sequence space. Then we have

(i) X is perfect $\Rightarrow$ X is normal $\Rightarrow$ X is monotone.

(ii) X is normal $\Rightarrow$ $X^\alpha = X^\gamma$

(iii) X is monotone $\Rightarrow$ $X^\alpha = X^\beta$

corollary 3.11 $bv(\Delta_v)$ *is not normal.*

Theorem 3.12 $C_1(\Delta_v)$ *is not separable.*

Proof Let A be the of all sequence $x_a, x_b, x_c, \dots$ where

$$x_a = (k+a)_k = (1+a, 2+a, \dots), \quad x_b = (k+b)_k = (1+b, 2+b, \dots), \dots$$

with $|a-b| > \frac{1}{2}$; $a, b \in \mathbb{R}$. Clearly, $A \subset C_1(\Delta_v)$ and A is uncountable. Let D be any dense set in $C_1(\Delta_v)$.

Define a map $f: A \to D$ as follows:

Let $x_a \in A \subset C_1(\Delta_v)$. As D is dense in $C_1(\Delta_v)$, so there exists some $z_{x_a} \in D$ such that $\|x_a - z_{x_a}\|_{\Delta_v} < \frac{1}{4}$.

We set $f(x_a) = z_{x_a}$.

For $x_a, x_b \in A$, we have

$$\begin{aligned} \|x_a - x_b\|_{\Delta_v} &= |(x_a - x_b)_1| + sup_k \frac{1}{k} |\sum_{i=1}^{k} \Delta\,(v_a x_a - v_b x_b)_i| \\ &= |(1+a) - (1+b)| + sup_k \frac{1}{k} |\sum_{i=1}^{k} \Delta\,(v_a x_a - v_b x_b)_i| \\ &\geq |a-b| \\ &> \frac{1}{2} \end{aligned}$$

Now

$$\begin{aligned} \|x_a - x_b\|_{\Delta_v} &\leq \|x_a - z_{x_a}\|_{\Delta_v} + \|z_{x_a} - x_b\|_{\Delta_v} \\ \|z_{x_a} - x_b\|_{\Delta_v} &\geq \|x_a - x_b\|_{\Delta_v} - \|x_a - z_{x_a}\|_{\Delta_v} \\ &> \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned}$$

and already we have $\|x_b - z_{x_b}\| < \frac{1}{4}$, therefore $z_{x_a} \neq z_{x_b}$. Hence f is one-one. As $f(A) \subset D$. So D is uncountable. Thus, $C_1(\Delta_v)$ has no countable dense set.

Corollary 3.13 $C_1(\Delta_v)$ does not have a Schauder basis.

Corollary 3.14 $C_1(\Delta_v)$ does not have the AK-property.

Theorem 3.15 $C_1(\Delta_v)$ is not normal and hence neither perfect nor convergence free.
Proof Taking $x = (x_k) = (k-1)$ and $y = (y_k) = (-1)^k(k-1)$, we see that $x \in C_1(\Delta_v)$ but $y \notin C_1(\Delta_v)$ although $|y_k| \leq |x_k|, k \geq 1$ and so $C_1(\Delta_v)$ is not normal. It is well known that every perfect space, and also every convergence free space, is normal and Consequently $C_1(\Delta_v)$ is neither perfect nor convergence free.

Theorem 3.16 $C_1(\Delta_v)$ is neither monotone nor a sequence algebra.
Proof Take $x = (x_k) = (k) \in C_1(\Delta_v)$. Consider $y = (y_k)$ where

$$y_k = \begin{cases} x_k, & \text{if k is odd} \\ 0, & \text{if k is even} \end{cases}$$

i.e, $y = (1,0,3,0,5,\dots)$. Then $(\Delta_v y_k) = (1,-3,3,-5,\dots)$ and $(\Delta_v y_k) \notin C_1$, i.e. $y_k \notin C_1(\Delta_v)$ and hence $C_1(\Delta_v)$ is not monotone. To see that $C_1(\Delta_v)$ is not sequence algebra, take $x = y = (k)$ and observe that $x, y \in C_1(\Delta_v)$ but $xy = (k^2) \notin C_1(\Delta_v)$.

Köthe - Toeplitz duals

Theorem 4.1

$$[C_1(\Delta_v)]^\alpha = \{a = (a_k) : \sum_k k\,|a_k| < \infty\} = D_1$$

Proof Let $a = (a_k) \in D_1$.
For any $x = (x_k) \in C_1(\Delta_v)$,
we have

$$\frac{1}{k}\left(\sum_{i=1}^{k} \Delta\, v_i x_i\right) \in c$$

i.e.,

$$\frac{1}{k}(v_1x_1 - v_{k+1}x_{k+1}) \in c$$

and so there exists some $M > 0$ such that $|x_k| \leq M(k-1) + x_1$ for $k \geq 1$ and hence $sup_k \frac{1}{k}|x_k| < \infty$, which implies that

$$\sum_k |a_k x_k| = \sum_k (k|a_k|)\,(k^{-1}|x_k|) < \infty$$

Thus, $a = (a_k) \in [C_1(\Delta_v)]^\alpha$.
Conversely, let $a = (a_k) \in [C_1(\Delta_v)]^\alpha$.
Then $\sum_k |a_k x_k| < \infty$ for all $x = (x_k) \in [C_1(\Delta_v)]^\alpha$. Taking $x_k = k$ for all $k \geq 1$, we have $x = (x_k) \in [C_1(\Delta_v)]^\alpha$. when $\sum_k k\,|a_k| < \infty$.
Similarly we can show that,

$$[l_\infty(\Delta_v)]^\alpha = \{a = (a_k): \sum_k k\,|a_k| < \infty\} = D_1$$

$$[c(\Delta_v)]^\alpha = \{a = (a_k): \sum_k k\,|a_k| < \infty\} = D_1$$

$$[c_0(\Delta_v)]^\alpha = \{a = (a_k): \sum_k k\,|a_k| < \infty\} = D_1$$

$$[bv(\Delta_v)]^\alpha = \{a = (a_k): \sum_k k\,|a_k| < \infty\} = D_1$$

i.e.,

$$[l_\infty(\Delta_v)]^\alpha = [c(\Delta_v)]^\alpha = [c_0(\Delta_v)]^\alpha = [bv(\Delta_v)]^\alpha = [C_1(\Delta_v)]^\alpha = D_1$$

So, we conclude that $\alpha - duals$ of difference sequence spaces $[l_\infty(\Delta_v)]$, $[c(\Delta_v)]$, $[c_0(\Delta_v)]$, $[bv(\Delta_v)]$and $[C_1(\Delta_v)]$ coincide.

Theorem 4.2

$$[C_1(\Delta_v)]^{\alpha\alpha} = \{a = (a_k): sup_k \frac{1}{k}|a_k| < \infty\} = D_2$$

The result follows in view of Theorem 4.1 and the fact ([10], Theorem 4.3) that $[C_1(\Delta)]^{\alpha\alpha} = D_2$

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