RESTRAINED EDGE DETOUR MONOPHONIC DOMINATION NUMBER OF A GRAPH

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Abstract :

In this paper the concept of *restrained edge detour monophonic domination number M of a graph* G is introduced. For a connected graph $G = (V, E)$ of order at least two, a *restrained edge detour monophonic dominating set M* of a graph G is a edge detour monophonic dominating set such that either $M = V$ or the sub graph induced by $V – M$ has no isolated vertices. The minimum cardinality of a restrained edge detour monophonic dominating set of G is the *restrained edge detour monophonic domination number* of G and is denoted by $\gamma_{edm_r}(G)$. We determine bounds for it and characterize graphs which realize these bounds. It is shown that for any two integers p and q with $2 \leq p \leq q$ there is a connected graph G such that $m(G) = p$ and $\gamma_{em}(G) = \gamma_{edm}(G) = \gamma_{edm_r}(G) = q$. If any two positive integers p and q such that $2 \leq p \leq q$ thereexists a connected graph G such that $\gamma_m(G) = \gamma_{em}(G) = p$ and $\gamma_{edm}(G) = \gamma_{edm_r}(G) = q$. If any three positive integers p, q and r with $2 < p < q < r$ thereexists a connected graph G such that $m(G) = p, em(G) = q$ and $\gamma_{edm}(G) = \gamma_{edm_r}(G) = r$. If any two positive integers p and q there is a connected graph G such that $\gamma(G) = p$ and $edm(G) = q$ with $\gamma_{edm}(G) = \gamma_{edm_r}(G) = p + q - 1$.

Keywords : Edge detour monophonic set, Edge detour monophonic number, edge detour monophonic dominating set, edge detour monophonic domination number, restrained edge detour monophonic dominating set and restrained edge detour monophonic domination number.

AMS Subject classification : 05C12.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [1] and [2]. The *neighborhood* of a vertex v of G is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v of G is an *extreme* vertex if the sub graph induced by its neighborhood is complete. A vertex v of G is an *semi-extreme* vertex of G if $\Delta(\langle N(v)\rangle) = |N(v)| - 1$. That is , the induced subgraph of $N(v)$ has a full vertex in $N(v)$. A vertex with degree one is called an *end vertex*. A vertex v of a connected graph G is called a *support vertex* of G if it is adjacent to an end vertex of G. A vertex v in a connected graph G is a *cut vertex* of G, if $G - v$ is disconnected. A *chord* of a path $u_1, u_2, u_3, \dots, u_k$ in G is an edge $u_i u_j$ with $j \geq i + 2$. A path P is called a *monophonic path* if it is a chordless path. A set M of vertices of G is a *monophonic set* of G if each vertex of G lies on $u - v$ monophonic path for some u and v in M. The minimum cardinality of a monophonic set M of G is the *monophonic number* of G and is denoted by $m(G)$ [5]. For a subset D of vertices, we call D is a dominating set for each $x \in V(G) - D$, x is adjacent to at least one vertex of D. The domination number of D is the minimum cardinality of a dominating set of G and is denoted by $\gamma_m(G)$ [3]. A longest $x-y$ monphonic path is called an $x-y$ detour monophonic path. An edge detour monophonic set of G is a set M of vertices such that every edge of G lies on an edge detour monophonic path joining some pair of vertices in M. The minimum cardinality of an edge detour monophonic set of G is the *edge detour monophonic number* of G and is denoted by $edm(G)$ [6]. A set of vertices M in G is called an edge detour monophonic dominating set if M is both edge detour monophonic set and a dominating set. The minimum cardinality of an edge detour monophonic dominating set of G is the *edge detour monophonic domination number* of G and is denoted by $\gamma_{edm}(G)$. A *restrained edge detour monophonic dominating set* of G is an edge detour monophonic dominating set M such that either $M = V$ or the sub graph induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained edge detour monophonic dominating set of G is the *restrained edge detour monophonic domination number* of G and is denoted by $\gamma_{edm_r}(G)$.

The following theorems will be used in the sequel.

Theorem 1.1. [6] Each semi-extreme vertex of a graph G belongs to every edge detour monophonic set of G. In particular, if the set M of all semi-extreme vertices of G is an edge detour monophonic set, then M is the unique minimum edge detour monophonic set of G.

Theorem 1.2. [6] Let G be a connected graph with cut – vertex v and let M be an edge detour monophonic set of G. Then every component of $G-v$ contains an element of M.

Theorem 1.3. [5] No cut vertex of a connected graph G belongs to any minimum monophonic set of G.

Theorem 1.4. [4] Each extreme vertex of a connected graph G belongs to every monophonic dominating set of G.

Theorem 1.5. [7] If T is a tree with k end vertices, then $dm(T) = k$.

Theorem 1.6. [6] For any graph G of order n, $2 \leq edm(G) \leq n$.

Theorem 1.7. [6] For the complete graph $K_n (n \geq 2)$, $edm(K_n) = n$.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Restrained Edge Detour Monophonic Domination Number of a Graph

Definition 2.1. A set $M \subseteq V(G)$ is termed *edge detour monophonic dominating* set if M is both an edge detour monophonic set and a dominating set. The minimum counting number among all edge detour monophonic dominating sets is called edge detour monophonic domination number and is denoted by γ_{edm}(G). A *restrained edge detour monophonic dominating set* M of a graph G is a edge detour monophonic dominating set such that either $M = V$ or the sub graph induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained edge detour monophonic dominating set of G is the *restrained edge detour monophonic domination number* of G and is designated by $\gamma_{edm_r}(G)$.

Example 2.2. In Figure 1, $M_1 = \{v_1, v_6, v_7\}$ is a minimum edge detour monophonic set. That is $edm(G) = 3$. The set $M_2 = \{v_1, v_3, v_6, v_7\}$ is a minimum edge detour monophonic dominating set. Thus $\gamma_{edm}(G) = 4$. Clearly the set $M_3 = \{v_1, v_2, v_3, v_6, v_7\}$ is a restrained edge detour monophonic dominating set. Therefore $\gamma_{edm_r}(G) = 5$.

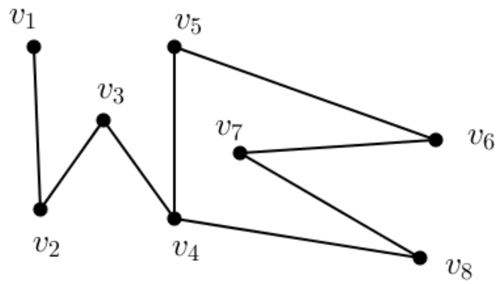

Figure 1: Graph G with $\gamma_{edm_r}(G) = 5$.

Theorem 2.3. For any graph G of order $n, 2 \leq \gamma_{edm_r}(G) \leq n$.

Proof : A restrained edge detour monophonic dominating set needs at least two vertices and so $\gamma_{edm_r}(G) \geq 2$. Clearly, the set of all vertices of G is an edge detour monophonic set of G so that $\gamma_{edm_r}(G) \leq n$.

Theorem 2.4. Each extreme vertex of a connected graph G belongs to every restrained edge detour monophonic dominating set of G. Moreover, if the set M of all extreme vertices of G is a restrained edge detour monophonic dominating set, then M is the unique minimum restrained edge detour monophonic dominating set of G.

Proof : Since every restrained edge detour monophonic dominating set of G is a edge detour monophonic dominating set of G, by Theorem 1.1, each extreme vertex belongs to every restrained edge detour monophonic dominating set of G. If the set M is the set of all extreme vertices of G, then $\gamma_{edm_r}(G) \geq |M|$. On the other hand , if M is a restrained edge detour monophonic dominating set of G, then $\gamma_{edm_r}(G) \leq |M|$. Therefore $\gamma_{edm_r}(G) = |M|$ and M is the unique minimum restrained edge detour monophonic dominating set of G.

In particular if G is an extreme graph then $V(G)$ is the unique minimum restrained edge detour monophonic dominating set of G. Note that the set of all extreme vertices need not form a restrained edge detour monophonic dominating set. Path graph with more than 5 vertices is an example.

Corollary 2.5. For the complete graph $K_p(p \geq 2), \gamma_{edm_r}(G) = p$.

Theorem 2.6. Let G be a connected graph with cut - vertices and let M be a restrained edge detour monophonic dominating set of G. If v is a cut - vertex of G then every component of $G - v$ contains an element of M.

Proof : Let A be a connected component $G - v$ that contains no vertices of M and let x be any vertex in A. Since M is a restrained edge detour monophonic dominating set, there is a pair of vertices p and q in M such that x present on a $p - q$ edge detour monophonic path $P: p = x_0, x_1, \ldots x, \ldots x_n = q$ in G with $x \neq p, q$. Then the $p - x$ subpath of P and $x - q$ sub path of P both contains v so that P is not a path. This is a contradiction.

Theorem 2.7. For a connected graph G of order p, $2 \leq edm(G) \leq \gamma_{edm}(G) \leq \gamma_{edm_r}(G) \leq p$.

Proof : Any edge detour monophonic dominating set of G needs at least two vertices and so that $edm(G) \geq 2$. Since every edge detour monophonic dominating set of G is also a edge detour monophonic set of G, it follows that $edm(G) \leq \gamma_{edm}(G)$. Also Since every restrained edge detour monophonic dominating set of G is also a edge detour monophonic dominating set of G, it follows that $\gamma_{edm}(G) \leq \gamma_{edm_r}(G)$. Since $V(G)$ is a restrained edge detour monophonic dominating set of G, it is clear that $\gamma_{edm_r}(G) \leq p$. Hence $2 \leq edm(G) \leq \gamma_{edm}(G) \leq \gamma_{edm_r}(G) \leq p$.

Corollary 2.8. Let G be a connected graph, if $\gamma_{edm_r}(G) = 2$, then $edm(G) = 2$.

For any non – trivial path of order at least 5, the edge detour monophonic number is 2 and the restrained edge detour monophonic domination number is 5. This shows that the converse of the Corollary 2.8 need not be true.

Remark 2.9. The bounds in Theorem 2.6 are sharp. For complete graph $G = K_2, \gamma_{edm_r}(G) = 2$ and for the complete graph $G = K_p, \gamma_{edm_r}(G) = p$. For the graph given in Figure 2, $M_1 = \{v_1, v_6, v_7\}$ is a minimum edge detour monophonic set of G. That is $edm(G) = 3$. The set $M_2 = \{v_1, v_3, v_6, v_7\}$ is a minimum edge detour monophonic dominating set of G. Thus $\gamma_{edm}(G) = 4$. $M_3 = \{v_1, v_2, v_3, v_6, v_7\}$ is a minimum restrained edge detour monophonic dominating set of G. Therefore $\gamma_{edm_r}(G) = 5$. Clearly order of the graph G is $p = 8$ so that $2 < edm(G) < \gamma_{edm}(G) < \gamma_{edm_r}(G) < p$. Hence all the parameters in Theorem 2.7 are distinct.

Theorem 2.10. Let G be a connected graph of order $p \geq 2$. Then, $G = K_2$ if and only if $\gamma_{edm_r}(G) = 2$.

Proof : If $G = K_2$, then $\gamma_{edm_r}(G) = 2$. Conversely, let $\gamma_{edm_r}(G) = 2$. Let $M = \{u, v\}$ be a minimum restrained edge detour monophonic dominating set of G. Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a $u - v$ restrained edge detour monophonic dominating path of G and so $G = K_2$.

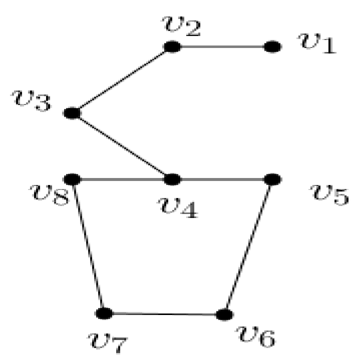

Figure 2

Corollary 2.11. For any nontrivial tree T of order p, $\gamma_{edm_r}(T) = p$.

Proof : It follows from Theorems 1.5 and 2.3.

Theorem 2.12. Each semi- extreme vertex belongs to every minimum restrained edge detour monophonic dominating set.

Proof: Let S be the set of all semi- extreme vertices of G Let M be any restrained edge detour monophonic dominating set of G. Suppose , take a vertex $x \in S$ such that $x \notin M$. Now $\Delta(\langle N(x)\rangle) = |N(x)| - 1$. Therefore there is a vertex $y \in N(x)$ such that $deg_{\langle N(x)\rangle}(y) = |N(x)| - 1$. On the other hand M is a restrained edge detour monophonic dominating set of G, the edge $e = xy$ present on a $u - v$ edge detour monophonic path $P : u = x_0, x_1, \ldots x_{i-1}, x_i = x, x_{i+1} = y, \ldots, x_n = v$ with $u, v \in M$. If $x \notin M$ then x is an internal vertex of the path P. Now $deg_{\langle N(x)\rangle}(y) = |N(x)| - 1$ so that y is adjacent to x_{i-1}. It contradicts that P is an edge detour monophonic path.

Theorem 2.13. Each extreme vertex belongs to every minimum restrained edge detour monophonic dominating set.

Proof: Since every extreme vertices are also semi- extreme vertices, the result follows from Theorem 2.12.

Remark 2.14. The set of all extreme vertices need not form a restrained edge detour monophonic dominating set. For the path graph P_n of $n > 4$ vertices, only the end vertices are extreme vertices, but they do not form a restrained edge detour monophonic dominating set.

Theorem 2.15. For the bipartite graph $K_{m,n}$, $\gamma_{edm_r}(G) = \begin{cases} 2, if\ m = n = 1 \\ min\{m,n\} + \max\{m,n\}\ if\ m,n \geq 2 \\ m+1,\ if\ m \geq 2, n = 1 \end{cases}$

Proof: (i) When $m = n = 1, K_{m,n} = K_2$, complete graph of two vertices. Hence $\gamma_{edm_r}(G) = 2$ (ii) When $m \geq 2, n = 1$ each m vertices are extreme vertices and belongs to every minimum restrained edge detour monophonic dominating set by Theorem 2.4. Since m vertices are extreme vertices and n is an isolated vertex. Therefore, $\gamma_{edm_r}(K_{m,n}) = m + 1$. (iii) For $m, n \geq 2$ assume $m \leq n$. Take $X = \{x_1, x_2, \ldots . x_m\}, Y = \{y_1, y_2, \ldots . y_n\}$ be a partition of G. Let $M = X$. Then M is a minimum edge detour monophonic dominating set in G. Also it is a minimum dominating set . Thus M is a minimum edge detour monophonic dominating set. Therefore $\gamma_{edm}(G) = |M| = m = min\{m,n\}$. Now $max\{m,n\}$ will be an isolated vertex. Therefore $\gamma_{edm_r}(G) = min\{m,n\} + \max\{m,n\} = m + n$.

Theorem 2.16. Let G be a connected graph of order n. If G has more than one vertex of degree $n-1$, then every restrained edge detour monophonic dominating set contains all vertices of degree $n-1$.

Proof: Let G be a graph of order n with more than one vertex of degree $n-1$ Suppose u and v are vertices having the degree $n-1$. Then the edge uv is not an edge of any edge detour monophonic path joining two vertices of G other than u and v. So both u and v belong to every edge detour monophonic set of G. That is, every vertex of degree $n-1$ belongs to every edge detour monophonic set. Obviously every vertex of degree $n-1$ belongs to every restrained edge detour monophonic dominating set.

The following results can be derived directly from the definition of restrained edge detour monophonic domination number and the above Theorems,

Result 2.17. For the Wheel graph W_n with n vertices, $\gamma_{edm_r}(W_n) = 4, n \geq 6$.

Result 2.18. For the Star graph $G = K_{1,n-1}$, $\gamma_{edm_r}(G) = n$.

3 Realization Results

Theorem 3.1 : For any two integers p and q with $2 \leq p \leq q$ there is a connected graph G such that $m(G) = p$ and $\gamma_{em}(G) = \gamma_{edm}(G) = \gamma_{edm_r}(G) = q$.

Proof : We prove this theorem for considering three cases.

Case 1: $p = q$.

Let H be a graph having vertices $v_1, v_2, , v_3, v_4, v_5$. Let G be a graph obtained from H by adding new vertices $q_1, q_2, \ldots \ldots, q_{p-2}$ to v_1. The graph G is shown in Figure 3. The set $M = \{q_1, q_2, \ldots \ldots, q_{p-2}\}$ is the set of all extreme vertices. Clearly every restrained edge detour monophonic dominating set of G contains M. The set $M_1 = M \cup \{v_4, v_5\}$ is a minimum monophonic set. Also it is an edge monophonic and edge monophonic dominating set. Thus $m(G) = \gamma_{em}(G) = p$. Clearly it is an edge detour monophonic dominating and restrained edge detour monophonic dominating $\gamma_{edm}(G) = \gamma_{edm_r}(G) = p$.

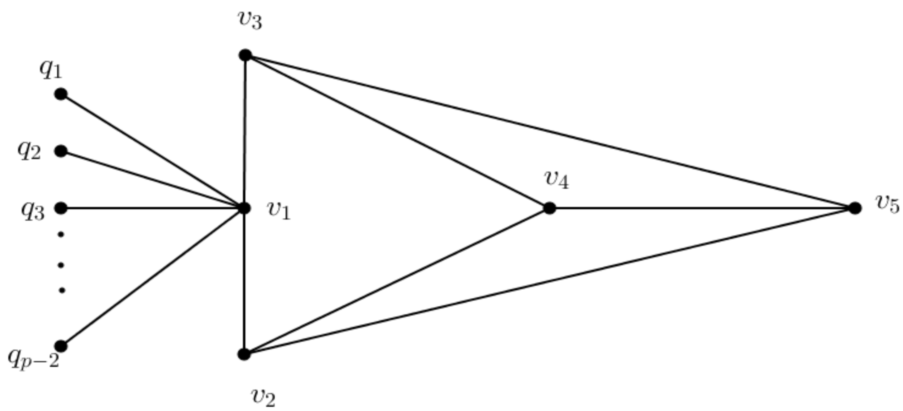

Figure 3

Case 2: $p = q - 1$. Let G be the graph as given in Figure 4. Here, the vertex set $M_2 = \{u_1, u_2, \dots \dots, u_{p-1}, v_5\}$ is the set of all extreme vertices and they form a monophonic set. Therefore $m(G) = p$. The set $M_3 = M_2 \cup \{v_3\}$ is a minimum edge monophonic dominating set and edge detour monophonic dominating set. Also it is the minimum restrained edge detour monophonic dominating set. Therefore, $\gamma_{em}(G) = \gamma_{edm}(G) = \gamma_{edm_r}(G) = p + 1 = q$.

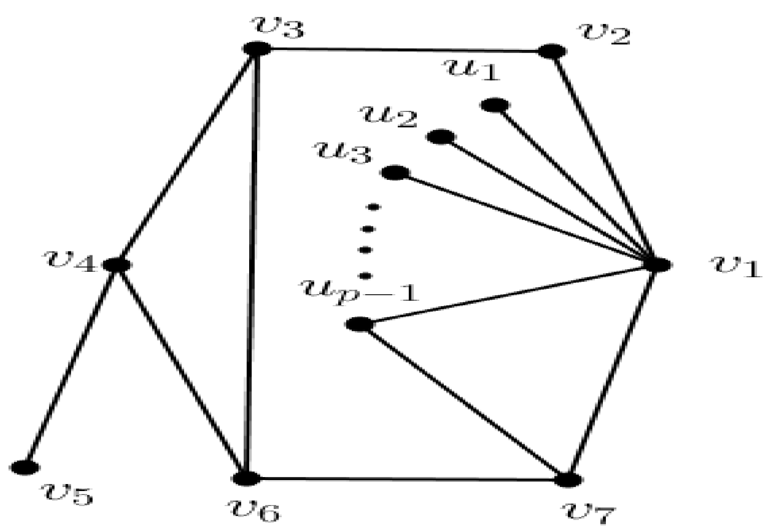

Figure 4

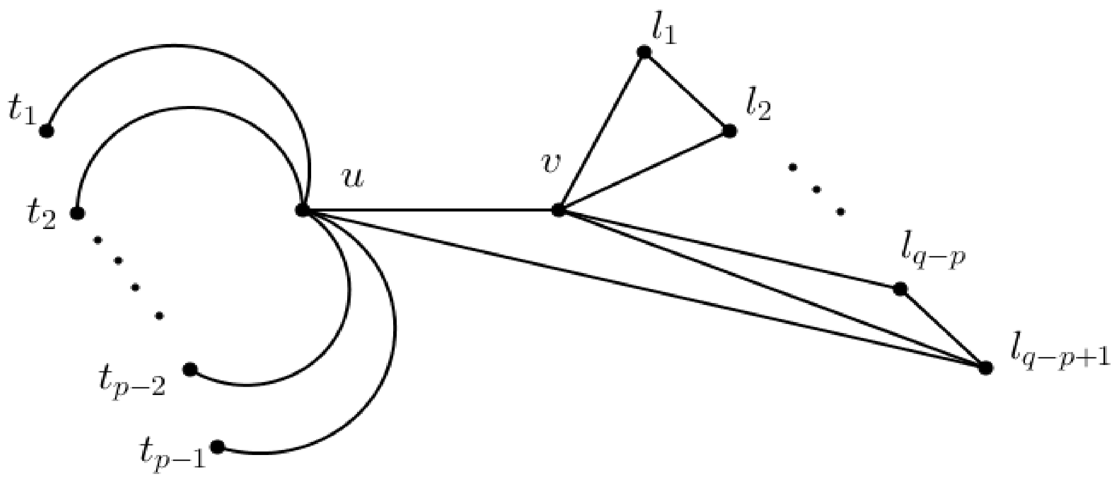

Figure 5

Case 3: $p \leq q - 2$. Take a set of disjoint vertices $M_4 = \{t_1, t_2, \dots \dots, t_{p-1}\}$and a path $P: l_1, l_2, \dots \dots, l_{q-p+1}$ with the set of vertices $M_5 = \{l_1, l_2, \dots \dots, l_{q-p+1}\}$. Let uv be an edge such that each t_i $(1 \leq i \leq p - 1)$ adjacent with u and each l_i $(1 \leq i \leq q - p + 1)$ adjacent with v. Furthermore u is incident with l_{q-p+1} . The graph G is given in Figure 5. Then $M_6 = M_5 \cup \{l_1\}$ is a minimum monophonic set. Therefore $m(G) = p$. The set $M_7 = M_4 \cup M_5$ is a minimum edge detour monophonic dominating set and restrained edge detour monophonic dominating set. Thus $\gamma_{em}(G) = \gamma_{edm}(G) = \gamma_{edm_r}(G) = (p - 1) + (q - p + 1) = q$.

Theorem 3.2 : If any two positive integers p and q such that $2 \leq p \leq q$ there exists a connected graph G such that $\gamma_m(G) = \gamma_{em}(G) = p$ and $\gamma_{edm}(G) = \gamma_{edm_r}(G) = q$.

Proof : Let $P_5: v_1, v_2, v_3, v_4, v_5$ be the path of order 5. Let $P_3: u_1, u_2, u_3$ be the path of order 3. Now introducing the new non adjacent vertices $k_1, k_2, \dots \dots, k_{q-p}, l_1, l_2, \dots \dots, l_{p-3}$. Let G be the graph obtained by joining each $k_i (1 \leq i \leq q - p)$ to v_4 and v_2 and joining each $l_i (1 \leq i \leq p - 3)$ to v_1. Also join u_1 to v_4 and u_3 to v_2. The graph G is shown in Figure 6.

Let $M_1 = \{l_1, l_2, \dots \dots, l_{p-3}, v_2, u_2, v_5\}$. Then M_1 is a minimum monophonic dominating set and edge monophonic dominating set. Hence $\gamma_m(G) = \gamma_{em}(G) = p$. It is easily verified that the set $M_2 = \{l_1, l_2, \dots \dots, l_{p-3}, v_5, k_1, k_2, \dots \dots, k_{q-p}, v_3, u_2\}$ is a minimum edge detour monophonic dominating set and restrained edge detour monophonic dominating set having $p - 3 + q - p + 3 = q$ vertices.

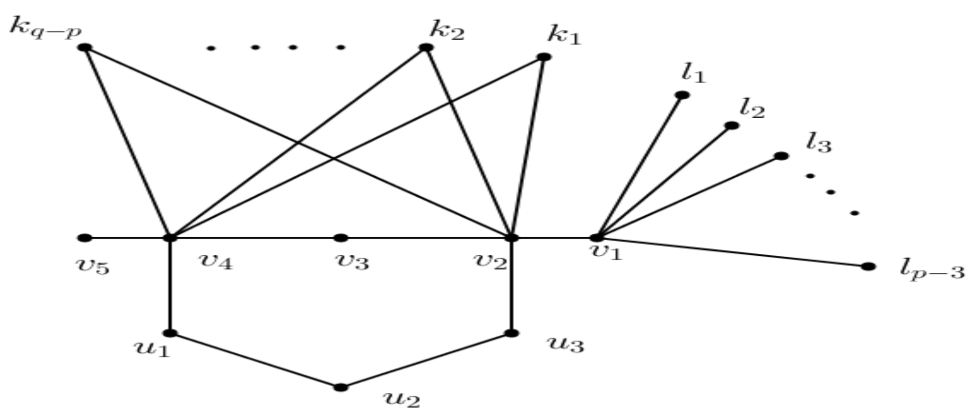

Figure 6

Theorem 3.3 : If any three positive integers p, q and r with $2 < p < q < r$ there exists a connected graph G such that $m(G) = p, em(G) = q$ and $\gamma_{edm}(G) = \gamma_{edm_r}(G) = r$.

Proof : We prove this theorem for considering three cases.

Case 1: $q = p + 2$. Inspect the graph given in Figure 7. Let $M_1 = \{l_1, l_2, \ldots \ldots, l_{p-1}, l_p\}$ is the set of all extreme vertices. Clearly it is a monophonic set and is minimum and so that $m(G) = p$. Take the set $M_2 = M_1 \cup \{z_2, z_4\}$. Clearly M_2 is a minimum edge monophonic set. Thus $em(G) = p + 2 = q$. Now the set $M_3 = M_2 \cup \{z_0, u_2, v_3, k_1, k_2, \ldots \ldots, k_{r-q-3}\}$ is a minimum edge detour monophonic dominating and restrained edge detour monophonic dominating set. Therefore $\gamma_{edm}(G) = \gamma_{edm_r}(G) = q + 3 + r - q - 3 = r$.

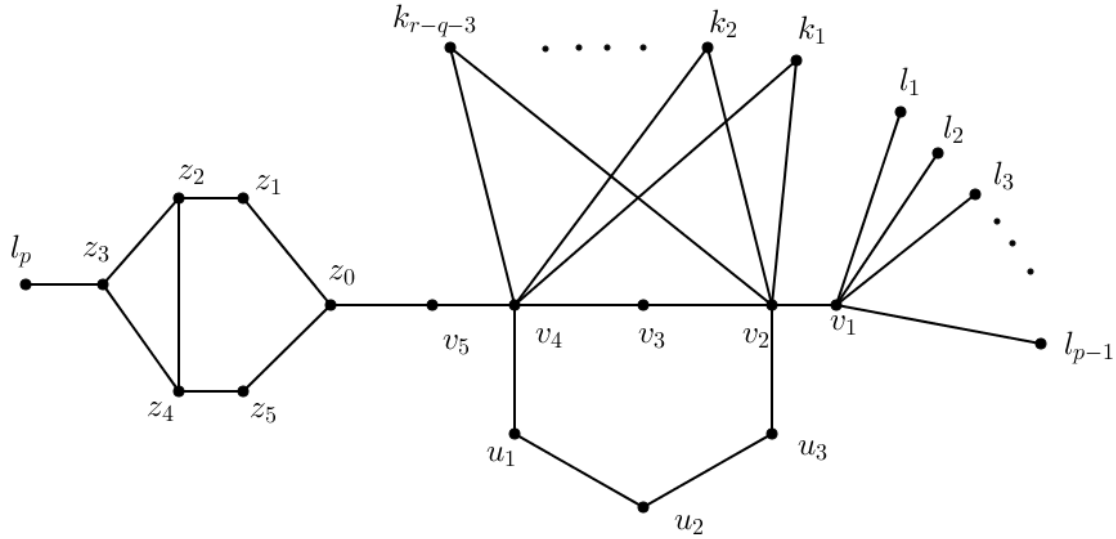

Figure 7

Case 2: $q \neq p + 2$. Study the graph given in Figure 8. Take $M_4 = \{z_1, z_2, \ldots \ldots, z_{p-1}, u_1\}$. It is easily verified that M_1 is a monophonic set and is minimum. Therefore $m(G) = p$. Let $M_5 = M_4 \cup \{u_1, u_2, \ldots \ldots, k_{q-p}\}$. Clearly it is a minimum edge monophonic set and so $em(G) = p + (q - p) = q$. Let $M_6 = M_5 \cup \{w_1, w_2, \ldots \ldots, w_{r-q-2}, x_2, v_3\}$ is a minimum edge detour monophonic dominating and restrained edge detour monophonic dominating set. Hence $\gamma_{edm}(G) = \gamma_{edm_r}(G) = q + (r - q - 2) + 2 = r$.

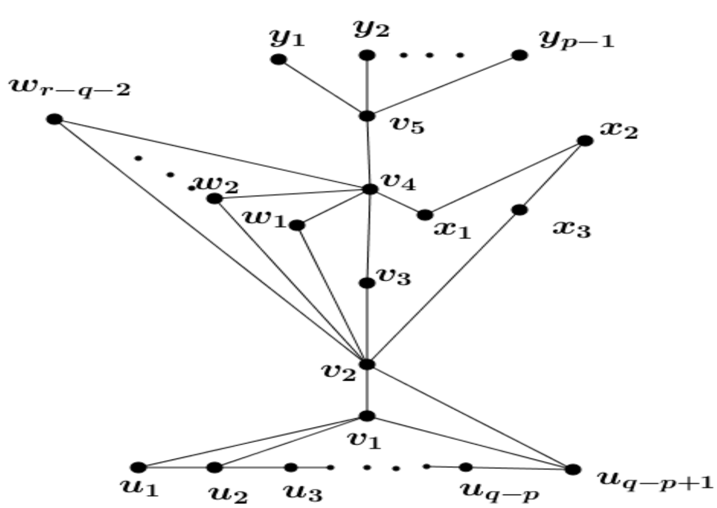

Figure 8

Theorem 3.4 : If any two positive integers p and q there is a connected graph G such that $\gamma(G) = p$ and $edm(G) = q$ with $\gamma_{edm}(G) = \gamma_{edm_r}(G) = p + q - 1$.

Proof : Let C_5be a cycle with vertex set.$\{u_1, u_2, u_3, u_4, u_5\}$ Join a path $P_{3p-6}: w_0, w_1, w_2, \dots, w_{3p-7}$ with the vertex u_3 . Join another set of non-adjacent vertices $\{v_1, v_2, \dots, v_{q-2}\}$ with u_1. The graph G is shown in Figure 9. The set $\{v_1, v_2, \dots, v_{q-2}, u_2, w_{3p-7}\}$ is a minimum edge detour monophonic set in G. Therefore $edm(G) = q$. The set $\{u_1, u_3, w_2,, w_5, \dots w_{3p-7}\}$ is a minimum dominating set in G. Therefore $\gamma(G) = p$. The set $\{v_1, v_2, \dots, v_{q-2}, u_1, u_3, u_2,, w_2,, w_5, \dots w_{3p-7}\}$ is a minimum edge detour monophonic dominating and restrained set edge detour monophonic dominating set having $p + q - 1$ vertices.

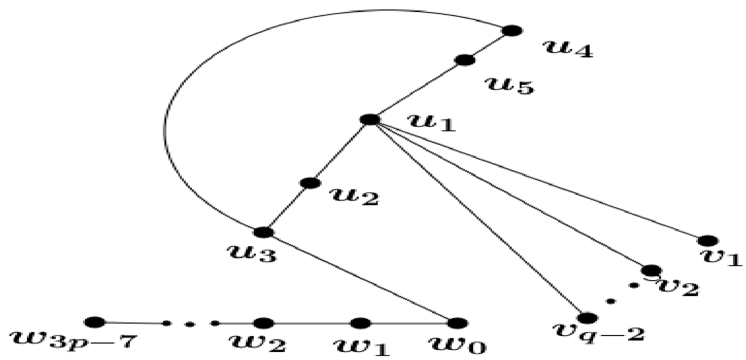

Figure 9

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