

On Anti Fuzzy Bi-ideal Of BCK-Algebra

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Abstract : The notion of fuzzy bi-ideals of a BCK-Algebra and the notion of anti-fuzzy bi-ideal of a BCK-Algebra are introduced. Using a collection of fuzzy bi-ideal we established anti-fuzzy bi-ideal of a BCK-Algebra . We show that μ is an anti-fuzzy bi-ideal in X iff μ^c is a fuzzy bi-ideal in X . The concept of the fuzzification of a anti-fuzzy bi-ideal and anti-fuzzy sub-algebra of a BCK-Algebra are introduced and investigate some of the properties.

IndexTerms – Fuzzy bi-ideal, Anti-Fuzzy Bi-ideal, Fuzzy sub-algebra, Anti-Fuzzy Sub-algebra.

I. INTRODUCTION

The concept of fuzzy sets was defined by Zadeh. In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-Algebras and BCI-Algebras. BCK-Algebras were studied by many researchers. In 1995, Jun applied the concept of fuzzy set to BCK-Algebras. He also got some interesting results. In 1990, Biswas introduced the concept of anti –fuzzy subgroup of group and now recently Hong and Jun, modifying Biswas idea, applied the concept of BCK-Algebras. So they defined the notion of anti-fuzzy ideal of a BCK-Algebras and obtain some useful result on it. To develop the theory of BCK-Algebras, the ideal theory plays an important role.

In this paper we introduce the notion of fuzzy bi-ideal and anti-fuzzy bi-ideal of a BCK-Algebra. Using a collection of fuzzy bi-ideal we established anti-fuzzy bi-ideal of a BCK-Algebra . We show that μ is an anti-fuzzy bi-ideal in X iff μ^c is a fuzzy bi-ideal in X . The concept of the fuzzification of a anti-fuzzy bi-ideal and anti-fuzzy sub-algebra of a BCK-Algebra are introduced and investigate some of the properties.

II. PRELIMINARIES

Definition 2.1:[3] By a BCK-algebra we mean a non-empty set X with a binary operation $*$ and a constant 0 satisfying the following condition:

- $((x*y)*(x*z)*(z*y)) = 0$
- $(x*(x*y))*y = 0$
- $x*x = 0$
- $0*x = 0$
- $x*y = 0$ and $y*x = 0$ imply that $x = y$ for all $x, y, z \in X$.

Definition 2.2:[1] A partial ordering “ \leq ” on X can be defined by $x \leq y$ iff $x*y = 0$.

Definition 2.3:[3] A non-empty subset S of a BCK-algebra X is called a sub -algebra of X if $x*y \in S$ whenever $x, y \in S$.

Definition 2.4:[3] A non-empty subset I of a BCK-algebra X is called an ideal of X if

- $0 \in I$
- $x*y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

Definition 2.5:[1] A fuzzy set μ in a non-empty set X we mean a function $\mu : X \rightarrow [0, 1]$.

Definition 2.6:[1] The complement of μ , denoted by $\bar{\mu}$ is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.7:[2] A fuzzy set μ in a BCK-algebra X is called a fuzzy sub-algebra of X if $\mu(x) \geq \min \{ \mu(x), \mu(y) \}$ for all $x, y \in X$.

Definition 2.8:[3] A fuzzy set μ in a BCK-algebra X is called a fuzzy ideal of X if

- $\mu(0) \geq \mu(x)$ for all $x \in X$
- $\mu(x) \geq \min \{ \mu(x*y), \mu(y) \}$ for all $x, y \in X$.

Definition 2.9 :[1] A mapping $f: X \rightarrow Y$ of a BCK-Algebra is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$.

Definition 2.10 :[3] A fuzzy subset μ of X is said to have sup property if for any subset T , there exists $t_0 \in T$ such that $\sup_{t \in T} \mu(t)$.

Definition 2.11 :[5] A family of fuzzy set $\{ \mu_i / i \in \Lambda \}$ is a BCK-Algebra X, the union $\bigvee_{i \in \Lambda} \mu_i$ of $\{ \mu_i / i \in \Lambda \}$ is defined by $\bigvee_{i \in \Lambda} \mu_i(x) = \sup \{ \mu_i / i \in \Lambda \}$ for each $x \in X$.

Definition 2.12 :[5] A family of fuzzy set $\{ \mu_i / i \in \Lambda \}$ is a BCK-Algebra X, the intersection $\bigwedge_{i \in \Lambda} \mu_i$ of $\{ \mu_i / i \in \Lambda \}$ is defined by $\bigwedge_{i \in \Lambda} \mu_i(x) = \inf \{ \mu_i / i \in \Lambda \}$ for each $x \in X$.

III. FUZZY BI-IDEAL OF BCK-ALGEBRA

Definition 3.1: A fuzzy subset μ in a BCK-algebra X is called fuzzy Bi-ideal if

- (a) $\mu(0) \geq \mu(x)$
- (b) $\mu(x * y) \geq \min \{ \mu(x * y * z), \mu(z) \}$ for all $x, y, z \in X$.

Example 3.2 : Consider a BCK-Algebra $X = \{ 0, x, y, z \}$ with the following Cayley table.

*	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	x	0	y
z	z	z	z	0

Define a fuzzy set $\mu : X \rightarrow [0,1]$ by $\mu(0) = 0.1, \mu(x) = 0.5, \mu(y) = 0.8, \mu(z) = 0.3$
Then by routine calculation it is easy to verify that μ is an anti fuzzy bi-ideal of a BCK-Algebra X.

IV. ANTI-FUZZY BI-IDEAL OF BCK-ALGEBRA

Definition 4.1 : A fuzzy subset μ in a BCK-Algebra X is called anti-fuzzy bi-ideal if $\mu(x * y) \leq \max \{ \mu(x * y * z), \mu(z) \}$ for all $x, y, z \in X$.

Example 4.2 : Consider a BCK-Algebra $X = \{ 0, a, b, c, d \}$ with the following Cayley table.

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	b	0	0
d	d	d	b	b	0

Define a fuzzy set $\mu : X \rightarrow [0,1]$ by $\mu(0) = 0.5, \mu(a) = 0.2, \mu(b) = 0.8, \mu(c) = 0.1, \mu(d) = 0.7$
Then by routine calculation it is easy to verify that μ is an anti-fuzzy bi-ideal of a BCK-Algebra X.

Theorem 4.3 : Let $f: X \rightarrow Y$ be an onto homomorphism of a BCK-Algebra X. If λ is a fuzzy bi-ideal in Y, then $f^{-1}(\lambda)$ is a fuzzy bi-ideal in a BCK-Algebra X.

Proof: Let λ be a fuzzy bi-ideal of Y.

$$\begin{aligned} &\text{For all } x, y, z \in X, \\ f^{-1}(\lambda)(x * y) &= \lambda(f(x * y)) \\ &= \lambda(f(x) * f(y)) \\ &\geq \min \{ \lambda(f(x) * f(y) * f(z)), \lambda(f(z)) \} \\ &= \min \{ f^{-1}(\lambda)(x * y * z), f^{-1}(\lambda)(z) \} \end{aligned}$$

Therefore, $f^{-1}(\lambda)(x * y) \geq \min \{ f^{-1}(\lambda)(x * y * z), f^{-1}(\lambda)(z) \}$.
Hence $f^{-1}(\lambda)$ is a fuzzy bi-ideal in a BCK-Algebra X.

Theorem 4.4 : Let X be a BCK-Algebra and μ be a fuzzy set in X. Then μ is an anti-fuzzy bi-ideal in X iff μ^c is a fuzzy bi-ideal in X.

Proof: Let X be a BCK-Algebra and μ be an anti-fuzzy bi-ideal in X.

For all $x, y, z \in X$,

$$\begin{aligned} \mu^c(x * y) &= 1 - \mu(x * y) \\ &\geq 1 - \max \{ \mu(x * y * z), \mu(z) \} \\ &= \min \{ 1 - \mu(x * y * z), 1 - \mu(z) \} \\ &= \min \{ \mu^c(x * y * z), \mu^c(z) \} \end{aligned}$$

Therefore, $\mu^c(x*y) \geq \min \{ \mu^c(x*y*z), \mu^c(z) \}$
Hence μ^c is a fuzzy bi-ideal in X.

Conversely suppose that μ^c is a fuzzy bi-ideal in X.
For all $x, y, z \in X$,

$$\begin{aligned} \mu(x*y) &= 1 - \mu^c(x*y) \\ &\leq 1 - \min \{ \mu^c(x*y*z), \mu^c(z) \} \\ &= \max \{ 1 - \mu^c(x*y*z), 1 - \mu^c(z) \} \\ &= \max \{ \mu(x*y*z), \mu(z) \} \end{aligned}$$

Therefore, $\mu(x*y) \leq \max \{ \mu(x*y*z), \mu(z) \}$.
Hence μ is an anti-fuzzy bi-ideal in X.

Theorem 4.5 : If $\{ \mu_i / i \in \Lambda \}$ is a family of anti-fuzzy bi-ideals of BCK-Algebra X and then $\bigvee_{i \in \Lambda} \mu_i$ is an anti fuzzy bi-ideal of a BCK-Algebra X.

Proof: Let $\{ \mu_i / i \in \Lambda \}$ be a family of anti-fuzzy bi-ideals of X and let $x, y, z \in X$.

Then we have ,

$$\begin{aligned} \bigvee_{i \in \Lambda} \mu_i(x*y) &= \sup \{ \mu_i(x*y) / i \in \Lambda \} \\ &\leq \sup \{ \max \{ \mu_i(x*y*z), \mu_i(z) \} / i \in \Lambda \} \\ &= \max \{ \sup \{ \mu_i(x*y*z) / i \in \Lambda \}, \sup \{ \mu_i(z) / i \in \Lambda \} \} \\ &= \max \{ (\bigvee_{i \in \Lambda} \mu_i)(x*y*z), (\bigvee_{i \in \Lambda} \mu_i)(z) \} \end{aligned}$$

Therefore $\bigvee_{i \in \Lambda} \mu_i(x*y) \leq \max \{ (\bigvee_{i \in \Lambda} \mu_i)(x*y*z), (\bigvee_{i \in \Lambda} \mu_i)(z) \}$

Hence $\bigvee_{i \in \Lambda} \mu_i$ is an anti-fuzzy bi-ideal of X.

Theorem 4.6 : If $\{ \mu_i / i \in \Lambda \}$ is a family of anti-fuzzy bi-ideals of BCK-Algebra X, then $\bigwedge_{i \in \Lambda} \mu_i$ is an anti fuzzy bi-ideal of a BCK-Algebras.

Proof : Proof is similar to thrm:4.5

Definition 4.7 : Anti Fuzzy Sub-algebra

Let X be a BCK-Algebra. Then anti-fuzzy sub-algebra of X if it satisfies $\mu(x*y) \leq \max \{ \mu(x), \mu(y) \}$ for all $x, y \in X$.

Theorem 4.8 : Let $\{ \mu_i / i \in I \}$ be any family of anti-fuzzy sub-algebra of X. Then $\bigcap_{i \in I} \mu_i$ and $\bigcup_{i \in I} \mu_i$ is an anti fuzzy sub-algebra of a BCK-Algebra X.

Proof : Let $\{ \mu_i / i \in I \}$ be a family of anti-fuzzy sub-algebra of X and let $x, y, z \in X$.

$$\begin{aligned} \bigcap_{i \in I} \mu_i(x*y) &= \inf \{ \mu_i(x*y) / i \in I \} \\ &\leq \inf \{ \max \{ \mu_i(x), \mu_i(y) \} / i \in I \} \\ &= \max \{ \inf \{ \mu_i(x) / i \in I \}, \inf \{ \mu_i(y) / i \in I \} \} \\ &= \max \{ (\bigcap_{i \in I} \mu_i)(x), (\bigcap_{i \in I} \mu_i)(y) \} \end{aligned}$$

Therefore, $\bigcap_{i \in I} \mu_i(x*y) \leq \max \{ (\bigcap_{i \in I} \mu_i)(x), (\bigcap_{i \in I} \mu_i)(y) \}$.

Hence $\bigcap_{i \in I} \mu_i$ is an anti fuzzy sub-algebra of a BCK-Algebra X.

Similarly, we can prove that $\bigcup_{i \in I} \mu_i$ is an anti fuzzy sub-algebra of X.

Theorem 4.9 : Let X be a BCK-Algebra. Then the fuzzy set μ is a fuzzy sub-algebra of a BCK-Algebra X iff μ^c is an anti fuzzy sub-algebra of a BCK-Algebra X.

Proof : Let μ be a fuzzy sub-algebra of X.

Let $x, y, z \in X$.

$$\begin{aligned} \mu^c(x*y) &= 1 - \mu(x*y) \\ &\leq 1 - \min \{ \mu(x), \mu(y) \} \\ &= \max \{ 1 - \mu(x), 1 - \mu(y) \} \\ &= \max \{ \mu^c(x), \mu^c(y) \} \end{aligned}$$

Therefore, $\mu^c(x*y) \leq \max \{ \mu^c(x), \mu^c(y) \}$

Conversely, let μ^c is an anti-fuzzy sub-algebra of X.

Let $x, y, z \in X$.

$$\begin{aligned}\mu(x*y) &= 1 - \mu^c(x*y) \\ &\geq 1 - \max\{\mu^c(x), \mu^c(y)\} \\ &= \min\{\mu(x), \mu(y)\}\end{aligned}$$

Therefore, $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$
Hence μ is a fuzzy sub-algebra of X .

Theorem 4.10: An anti-homomorphic pre-image of a fuzzy bi-ideal is a fuzzy bi-ideal of a BCK-Algebra X .

Proof: Let $f: X \rightarrow Y$ be an anti homomorphism of a BCK-Algebra X .

Let λ be a fuzzy bi-ideal of Y .
Let $x, y, z \in X$.

$$\begin{aligned}f^{-1}(\lambda)(x*y) &= \lambda(f(x*y)) \\ &= \lambda(f(x)*f(y)) \\ &\geq \min\{\lambda(f(x*y*z)), \lambda(f(z))\} \\ &= \min\{f^{-1}(\lambda)(x*y*z), f^{-1}(\lambda)(z)\}\end{aligned}$$

Therefore, $f^{-1}(\lambda)(x*y) \geq \min\{f^{-1}(\lambda)(x*y*z), f^{-1}(\lambda)(z)\}$
Hence $f^{-1}(\lambda)$ is a fuzzy bi-ideal of a BCK-Algebra X .

Theorem 4.11: An anti-homomorphic image of a fuzzy bi-ideal of a BCK-Algebra X with sup property is a fuzzy bi-ideal of a BCK-Algebra X .

Proof : Let $f: X \rightarrow Y$ be an anti-homomorphism of a BCK-Algebra X .

Let μ be a fuzzy bi-ideal with sup property.
Let λ be the image of μ .

Let $f(x), f(y), f(z) \in f(X)$ and

let $x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y)), z_0 \in f^{-1}(f(z))$ such that $\mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t)$,
 $\mu(y_0) = \sup_{t \in f^{-1}(f(y))} \mu(t), \mu(z_0) = \sup_{t \in f^{-1}(f(z))} \mu(t)$.

$$\begin{aligned}\lambda(f(x)*f(y)) &= \lambda(f(x*y)) \\ &= \sup_{t \in f^{-1}(f(x*y))} \mu(t) \\ &\geq \mu(x_0, y_0) \\ &\geq \min\{\mu(x_0*y_0*z_0), \mu(z_0)\} \\ &= \min\left\{\sup_{t \in f^{-1}(f(x*y*z))} \mu(t), \sup_{t \in f^{-1}(f(z))} \mu(t)\right\} \\ &= \min\{\lambda(f(x*y*z)), \lambda(f(z))\}\end{aligned}$$

Therefore $\lambda(f(x)*f(y)) \geq \min\{\lambda(f(x*y*z)), \lambda(f(z))\}$
Hence λ is a fuzzy bi-ideal of Y .

Theorem 4.12: An anti homomorphic pre-image of an anti-fuzzy bideal of a BCK-Algebra X is an anti-fuzzy bi-ideal of a BCK-Algebra X .

Proof: Similar proof of thrm 4.10

Theorem 4.13: An anti homomorphic image of an anti-fuzzy bideal with sup property of a BCK-Algebra X is an anti-fuzzy bi-ideal of a BCK-Algebra X .

Proof: Similar proof of thrm 4.11

Theorem 4.14: Let μ be a fuzzy set of a BCK-Algebra X , then μ is a fuzzy bi-ideal of X iff μ^c is an anti-fuzzy bi-ideal of a BCK-Algebra X .

Proof: It can be easily verified.

REFERENCES

- [1] L.A.Zadeh, "Fuzzy Sets, Information and Control ," 8(1965), 338–353.MR 36# 2509. Zbl 139. 24606.
- [2] Y.B.Jun & E.H.Roh," Fuzzy Commutative ideals of Bck-algebras, Fuzzy Sets & Systems 64(1994), no.3, 401-405, MR 95e:06051,zbl 846.06011.
- [3] Y.B.Jun, S.M.Hong, S.J.Kim, and S.Z.Song," Fuzzy ideals and Fuzzy Subalgebras of Bck-algebras", J.Fuzzy Math.7(1999),n0.2, 411-418, MR 2000c:06040,zbl 943.06010 .
- [4] Y.B.Jun, " A Note on Fuzzy Ideals in Bck-algebras", Math.Japon. 42(1995),no.2, 333-335,CMP 1356 395, zbl 834.06018.
- [5] M.Himaya Jaleela Begum," On anti fuzzy bi-ideals in near rings", International journal of Mathematics and soft computing,Vol.5, No.2.(2015), 75-82.

