DOUBT FUZZY BI - IDEAL OF BS-ALGEBRAS

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Abstract : In this paper, a new notion called BS-algebras, which is a generalization of the idea of BE/B/BCK, is introduced. We extend fuzzy ideal into fuzzy bi-ideal in BS-algebras. Then we introduce the notion of doubt fuzzy bi-ideal of BS-algebras. We investigate some algebraic nature of doubt fuzzy bi-ideal of BS-algebras. Doubt fuzzy bi-ideal of BS-algebras is also applied in Cartesian product. Finally, the homomorphic behaviour of doubt fuzzy bi-ideal of BS-algebras have been obtained.

***IndexTerms* –Fuzzy bi-ideal,Doubt fuzzy bi-ideal,Cartesian product,homomorphism(BS-algebras).**

I. INTRODUCTION

After the introduction of fuzzy subsets by L.A.Zadeh[3], several researches explored on the generalization of the notion of fuzzy subset. In 1966, Imai and Iseki introduced two classes of abstract algebras viz. BCK-algebras and BCI-algebras[1]. The class of BCK-algebras is a proper subclass of the class of BCI-algebras. J.Neggers and H.S. Kim introduced the notion of B-algebras[2] which is a generalisation of BCK-algebras. We introduce the notion of BS-algebras which is a generalisation of B-algebras. In this paper, we presented doubt fuzzy bi-ideal called DF bi-ideal of BS-algebras and establish some of their basic properties. The Cartesian product of fuzzy bi-ideal for BS-algebras has been introduced and some important properties are also studied. Finally, we investigate how to deal with homomorphism.

II. PRELIMINARIES

Definition: A BS-algebra is a non empty set with a constant 1 and a binary operation * satisfying the following axioms

(i) $x*x=1$
(ii) $x*1=x$
(iii) $1*x=x$
(iv) $(x*y)*z = x*(z*(1*y))\ \forall\, x, y, z \in X$

Note: We call X as BS-algebra. A binary relation $\leq$ on X can be defined by $x\leq y$ if and only if $x*y=1$.

Example: Let $X=\{1,a,b,c\}$ be a set with the following Cayley table

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X,*,1)$ is a BS-algebra.

Definition: Let μ be a fuzzy set in a BS-algebra X. Then μ is called a fuzzy sub algebra of X if

$$\mu(x*y) \geq \min\{\mu(x),\mu(y)\}\ \forall\, x, y \in X$$

Definition: A fuzzy set μ of BS-algebra X is called a fuzzy ideal of X if it satisfies the following conditions

(i) $\mu(1) \geq \mu(x)$
(ii) $\mu(y) \geq \min\{\mu(x),\mu(y*x)\}\ \forall\, x, y, z \in X$

Definition: A fuzzy subset μ in a BS-algebra X is called fuzzy bi-ideal if

(i) $\mu(1) \geq \mu(x)$
(ii) $\mu(y*z) \geq \min\{\mu(x), \mu(x*(y*z))\}\ \forall\, x, y, z \in X$

Definition: A fuzzy set μ of a BS-algebra X is called a doubt fuzzy sub algebra of X if $\mu(x*y) \le \max\{\mu(x),\mu(y)\}$ ∀ x, y ∈ X

III. DOUBT FUZZY BI IDEAL

Definition: A fuzzy set μ of BS-algebra X is called a doubt fuzzy (DF) ideal of X if

(i) $\mu(1) \le \mu(x)$

(ii) $\mu(y) \le \max\{\mu(x), \mu(y*x)\}$ ∀ x, y ∈ X

Definition: A fuzzy set μ of BS-algebra X is called a doubt fuzzy bi- ideal of X if

(i) $\mu(1) \le \mu(x)$

(ii) $\mu(y*z) \le \max\{\mu(x), \mu(x*(y*z))\}$ ∀ x, y, z ∈ X

Example: Let X= {1,a,b,c} be a set with the following Cayley table

*	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then (X, *, 1) is a BS-algebra. Define a fuzzy set $\mu:X\to[0,1]$ by $\mu(1)=\mu(a)=0.7$ and $\mu(b)=\mu(c)=0.9$. Then μ is a DF bi-ideal of X.

Theorem 1: Let μ be a DF bi-ideal of BS-algebra X. If $x \le y*z$ then $\mu(y*z) \le \mu(x)$ (i. e) μ is order-reversing.

Proof

Let μ be a DF bi-ideal of BS-algebra X.

Let $x\le y*z$ then $x*(y*z)=1$

For any x, y, z ∈ X

Now, $\mu(y*z) \le \max\{\mu(x),\mu(x*(y*z))\}$

$= \max\{\mu(x),\mu(1)\}$

Therefore, $\mu(y*z) \le \mu(x)$

Theorem 2: Let μ be a DF bi-ideal of BS-algebra X. If $x\le y*z$ then $\mu(y) \le \max\{\mu(x), \mu(z)\}$ ∀ x, y, z ∈ X

Proof

Let μ be a DF bi-ideal of BS-algebra X. For any x, y, z ∈ X

Now, $\mu(y) \le \max\{\mu(y*z),\mu(z)\}$

$\le \max\{\max\{\mu(x*(y*z)),\mu(x)\}, \mu(z)\}$

$= \max\{\max\{\mu(1),\mu(x)\}, \mu(z)\}$

$= \max\{\mu(x), \mu(z)\}$

Therefore, $\mu(y) \le \max\{\mu(x), \mu(z)\}$

Theorem 3: Let μ be a DF bi-ideal of BS-algebra X. If $\mu(x*(y*z))=\mu(1)$ then $\mu(y*z)\le\mu(x)$

Proof

Let μ be a DF bi-ideal of BS-algebra X. For any x, y, z ∈ X

Now, $\mu(y*z) \le \max\{\mu(x), \mu(x*(y*z))\}$

$= \max\{\mu(x), \mu(1)\}$

$= \mu(x)$

Therefore, $\mu(y*z) \le \mu(x)$

Theorem 4: Let μ_1 and μ_2 be two DF bi-ideal of BS-algebra X. Then $\mu_1 \cup \mu_2$ is also a DF bi-ideal of X.

Proof

Let μ be a DF bi-ideal of BS-algebra X. For any x, y, z ∈ X

(i) $(\mu_1\cup \mu_2)(1) = \max\{ \mu_1(1), \mu_2(1)\}$

$\le \max\{ \mu_1(x), \mu_2(x)\}$

$= (\mu_1\cup \mu_2)(x)$

Therefore, $(\mu_1\cup \mu_2)(1) \le (\mu_1\cup \mu_2)(x)$

(ii) $(\mu_1\cup \mu_2)(y*z) = \max\{ \mu_1(y*z), \mu_2(y*z)\}$

$\le \max\{\max\{ \mu_1(x), \mu_1(x*(y*z))\}, \max\{ \mu_2(x), \mu_2(x*(y*z))\}\}$

$=\max\{\max\{ \mu_1(x), \mu_2(x)\}, \max\{ \mu_1(x*(y*z)), \mu_2(x*(y*z))\}\}$

$=\max\{(\mu_1\cup \mu_2)(x), (\mu_1\cup \mu_2)(x*(y*z))\}$

$(\mu_1\cup \mu_2)(y*z) \le \max\{(\mu_1\cup \mu_2)(x), (\mu_1\cup \mu_2)(x*(y*z))\}$

Therefore, $\mu_1\cup \mu_2$ is a DF bi-ideal of X.

The above theorem can be generalised as

Theorem 5: Let $\{\mu_i/i=1,2,.....\}$ be a family of DF bi-ideal of BS-algebra X. Then $\cup_i \mu_i$ is also a DF bi-ideal of X, where $\cup_i \mu_i = \max\{\mu_i(x)/ i=1,2,.......\}$

Proof

Let μ be a DF bi-ideal of X. For any $x, y, z \in X$

(i) Now, $(\cup_i \mu_i)(1) = \sup\{\mu_i(1):i=1,2,.......\}$

$\leq \sup\{\mu_i(x):i=1,2,.......\}$

$=(\cup_i \mu_i)(x)$

Therefore, $(\cup_i \mu_i)(1) \leq (\cup_i \mu_i)(x)$

(ii) Now, $(\cup_i \mu_i)(y*z) = \sup\{\mu_i(y*z)/i=1,2,.......\}$

$\leq \sup\{\max\{\mu_i(x), \mu_i(x*(y*z))\}/i=1,2,.......\}$

$= \max\{\sup\{\mu_i(x)/i=1,2,.......\}, \sup\{\mu_i(x*(y*z)/i=1,2,.......\}$

$= \max\{(\cup_i \mu_i)(x), (\cup_i \mu_i)(x*(y*z))\}$

Therefore, $(\cup_i \mu_i)(y*z) \leq \max\{(\cup_i \mu_i)(x), (\cup_i \mu_i)(x*(y*z))\}$

Theorem 6: A fuzzy subset μ of BS-algebra X is a fuzzy bi-ideal of X iff its complement μ^c is DF bi-ideal of X.

Proof

Let μ be a fuzzy bi-ideal of BS-algebra X. For any $x, y, z \in X$

(i) $\mu^c(1) = 1- \mu(1)$

$\leq 1- \mu(x)$

$= \mu^c(x)$

Therefore, $\mu^c(1) \leq \mu^c(x)$

(ii) $\mu^c(y*z)=1- \mu(y*z)$

$\leq 1-\min\{\mu(x*(y*z)), \mu(x)\}$

$=\max\{\mu^c(x), \mu^c(x*(y*z))\}$

Therefore, $\mu^c(y*z) \leq \max\{\mu^c(x), \mu^c(x*(y*z))\}$

Conversely,

(i) $1-\mu(1)\leq 1-\mu(x)$

$-\mu(1) \leq -\mu(x)$

Therefore, $\mu(1) \geq \mu(x)$

(ii) $1-\mu(y*z) \leq \max\{1-\mu(x), 1-\mu(x*(y*z))\}$

$\leq 1-\min\{\mu(x), \mu(x*(y*z))\}$

Therefore, $\mu(y*z) \geq \min\{\mu(x),\mu(x*(y*z))\}$

Theorem 7: Let μ be a fuzzy subset of a BS-algebra X. If μ is a DF bi-ideal of X, then the lower level cut μ_t is an ideal of X for all $t \in [0,1], t > \mu(1)$

Proof

Let μ be a DF bi-ideal of BS-algebra X. For any $x, y, z \in X$

Let $x, y \in \mu_t$

Since $\mu(1) \leq \mu(x) \leq t \Rightarrow 1 \in \mu_t \quad \forall t \in [0,1]$

Again let $x, x*(y*z) \in \mu_t$

Therefore $\mu(x) \leq t$, $\mu(y*z) \leq t$

Now, $\mu(y*z) \leq \max\{\mu(x), \mu(x*(y*z))\}$

$\leq \max\{t, t\}=t$

Therefore, $\mu(y*z) \leq t \Rightarrow y*z \in \mu_t$

Hence, μ_t is an ideal of X

Theorem 8: Let μ be a DF bi-ideal of BS-algebra X. Then two lower level cuts μ_{t_1}, μ_{t_2} where $(t_1 < t_2)$ of μ are equal iff there is no $x \in X$ such that $t_1 < \mu(x) < t_2$

Proof

Let μ be a DF bi-ideal of X. For any $x, y, z \in X$

$\mu_t = \{x \in X/ \mu(x) \leq t\}$

Let $\mu_{t_1} = \mu_{t_2}$ where $(t_1 < t_2)$ and there exists $x \in X$ such that $t_1 < \mu(x) < t_2$

Take $\mu_{t_1} \subset \mu_{t_2}$, then $x \in \mu_{t_2}$ but $x \notin \mu_{t_1}$ which contradicts the fact that $\mu_{t_1} = \mu_{t_2}$

Hence there is no $x \in X$ such that $t_1 < \mu(x) < t_2$

Conversely, suppose that there is no $x \in X$ such that $t_1 < \mu(x) < t_2$

Therefore $\mu_{t_1} \subset \mu_{t_2}$ (since $t_1 < t_2$)

Again if x ∈ μ_{t_2} then μ(x) ≤ t_2 and by hypothesis we get μ(x) ≤ t_1 => $\mu_{t_2} \subset \mu_{t_1}$
Hence $\mu_{t_1} = \mu_{t_2}$

Definition: [4] Let μ_1 and μ_2 be two fuzzy sets of X. Then the intersection of μ_1 and μ_2 is denoted by $\mu_1 \cap \mu_2$ and is given by $\mu_1 \cap \mu_2 = \min\{\mu_1(x), \mu_2(x)\}$

Theorem 9: If {μ_i /i=1,2,.......} is a family of DF bi-ideal of BS-algebra X, then {∩μ_i /i=1,2,.......} is a DF bi-ideal

Proof

Let μ be a DF bi-ideal of BS-algebra X. For any x, y, z ∈ X

(i) $(\cap\mu_i)(1) = \inf\{\mu_i(1) /i=1,2,.......\}$

$\leq \inf\{\mu_i(x) /i=1,2,.......\}$
$= (\cap\mu_i)(x), i=1,2,.....$

Therefore, $(\cap\mu_i)(1) \leq (\cap\mu_i)(x), i=1,2,.....$

(ii) $(\cap\mu_i)(y*z) = \inf\{\mu_i(y*z) /i=1,2,.......\}$

$\leq \inf\{\max\{\mu_i(x), \mu_i(x*(y*z))\} /i=1,2,.......\}$
$= \max\{\inf\{\mu_i(x)/i=1,2,..\}, \inf\{\mu_i(x*(y*z)) /i=1,2,.......\}\}$
$= \max\{(\cap\mu_i)(x), (\cap\mu_i)(x*(y*z))/i=1,2,.....\}$

Therefore, $(\cap\mu_i)(y*z) \leq \max\{(\cap\mu_i)(x), (\cap\mu_i)(x*(y*z))/i=1,2,.....\}$

Hence {($\cap\mu_i$)/i=1,2,.....} is a DF bi-ideal of X.

4. PRODUCT OF DF BI IDEALS OF BS-ALGEBRAS

Definition: Let μ_1 and μ_2 be two DF bi-ideals of BS-algebra X. Then their Cartesian product is defined by $(\mu_1 \times \mu_2)(x,y)=\max\{\mu_1(x), \mu_2(y)\}$ Where $(\mu_1 \times \mu_2)$:X x X → [0,1] ∀ x, y ∈ X.

Theorem 10: Let μ_1 and μ_2 be two DF bi-ideals of BS-algebra X. Then $\mu_1 \times \mu_2$ is also a DF bi-ideal of X x X.

Proof

Let μ_1 and μ_2 be two DF bi-ideals of BS-algebra X. For any (x, y) ∈ X x X

We have $(\mu_1 \times \mu_2)(1,1) = \max\{\mu_1(1), \mu_2(1)\}$
$\leq \max\{\mu_1(x), \mu_2(y)\}$
$= (\mu_1 \times \mu_2)(x,y)$

Therefore, $(\mu_1 \times \mu_2)(1,1) \leq (\mu_1 \times \mu_2)(x,y)$

Again let $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ ∈ X x X

Then $(\mu_1 \times \mu_2)((y_1*z_1), (y_2*z_2)) = \max\{\mu_1(y_1*z_1), \mu_2(y_2*z_2)\}$
$\leq \max\{\max\{\mu_1(x_1), \mu_1(x_1*(y_1*z_1))\}, \max\{\mu_2(x_2), \mu_2(x_2*(y_2*z_2))\}\}$
$= \max\{\max\{\mu_1(x_1), \mu_2(x_2)\}, \max\{\mu_1(x_1*(y_1*z_1)), \mu_2(x_2*(y_2*z_2))\}\}$
$= \max\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(x_1*(y_1*z_1), x_2*(y_2*z_2))\}$

Therefore, $(\mu_1 \times \mu_2)((y_1*z_1), (y_2*z_2)) \leq \max\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(x_1*(y_1*z_1), x_2*(y_2*z_2))\}$

5. HOMOMORPHISM ON BS-ALGEBRAS

Definition: Let X and Y be two BS-algebras. A mapping f: X→Y is said to be homomorphism if f(x*y) = f(x)*f(y) ∀ x, y ∈ X

Note: Let X and Y be two BS-algebras and f: X→Y be a homomorphism. Then f(1) =1

Proof

Let X and Y be two BS-algebras

Let x ∈ X therefore f(x) ∈ Y

Now f(1)=f(x*x)=f(x)*f(x)=1*1 =1

Theorem 12: Let f: X→Y be an epimorphism of BS-algebras if λ be a DF bi-ideal of Y, then the pre image of λ under f is also a DF bi-ideal of X.

Proof

Let f: X→ Y be an epimorphism of BS-algebras if λ be a DF bi-ideal of Y.

Let μ be the pre image of λ under f then λ (f(x)) = μ(x) ∀ x ∈ X

Now, $\lambda(1^f) = \lambda(f(1)) \leq \lambda(f(x)) = \mu(x)$

$\mu(1) = \lambda(1^f) \leq \mu(x) \Rightarrow \mu(1) \leq \mu(x)$ ∀ x ∈ X

Again, $\mu(y*z) = \lambda(f(y*z)) \leq \max\{\lambda(f(x)), \lambda(f(x*(y*z)))\}$
$= \max\{\mu(x), \mu(x*(y*z))\}$

Therefore, $\mu(y*z) \leq \max\{\mu(x), \mu(x*(y*z))\}$ Which is true for all x, y, z ∈ X.

Hence μ is a DF bi-ideal of X.

Theorem 13: Let f: X→Y be an epimorphism where X and Y are two BS-algebras if λ be a fuzzy subset of Y such that $f^{-1}(\lambda)$ is DF bi-ideal of X, then λ is also a DF bi-ideal of Y.

Proof

Let X and Y are two BS-algebras if λ be a fuzzy subset of Y.

Let u, v, w ∈ X therefore there exists x, y, z ∈ X such that f(x) = u, f(y) = v, f(z) = w

Let μ be the pre image of λ under f then λ (f(x)) = μ(x) ∀ x ∈ X

Since μ is DF bi-ideal of X, $\mu(1) \leq \mu(x) \Rightarrow \lambda(f(1)) \leq \lambda(f(x))$

$\Rightarrow \lambda(1) \leq \lambda(u)\ \forall u \in X$

Again, $\mu(y*z) \leq \max\{\mu(x), \mu(x*(y*z))\}$ for all x, y, z ∈ X.

$\Rightarrow \lambda(f(y*z)) \leq \max\{\lambda(f(x)), \lambda(f(x*(y*z)))\}$

$\Rightarrow \lambda(f((y)*f(z))) \leq \max\{\lambda(f(x)), \lambda(f(x*(y*z)))\}$

$= \max\{\lambda(f(x)), \lambda(f(x)*f(y*z)))\}$

$= \max\{\lambda(f(x)), \lambda(f(x)*(f(y)*f(z)))\}$

$= \max\{\lambda(u), \lambda(u*(v*w))\}$

Therefore, $\lambda(v*w) \leq \max\{\lambda(u), \lambda(u*(v*w))\}$

Hence λ is a DF bi-ideal of Y.

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