A CRITICAL STUDY OF BRAHMAGUPTA'S THEOREMS ON CYCLIC QUADRILATERAL

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Abstract **:** In geometry, Brahmagupta's formula calculates the area enclosed by a cyclic quadrilateral. In this paper we have discussed about the two theorems of Brahmagupta and examine critically these two cases.

Keywords : Brahmagupta's theorem, Brahmagupta's formula, Ptolemy's theorem, cyclic quadrilateral, diagonal.

INTRODUCTION

Brahmagupta's most important contributions to the geometry are the two theorems of cyclic quadrilateral. In this paper we discuss about the two theorems of Brahmagupta and examine critically these two cases.

In ancient times, different Indian Mathematicians have given the rules to find the area of Triangle and Quadrilateral, but among all the Ancient Indian Mathematicians, the rules of Brahmagupta resemble to the modern mathematics.

BRAHMAGUPTA'S THEOREM 1.

स्थूलफलं त्रिचतुर्भुजबाहुप्रतिबाहु योगदलधातः।
भुजयोगार्धचतुष्टयभुजोनधातात् पदं सूक्ष्मम्।।

(Br. Sp. Si XII. 21)

(The gross area of a triangle or a quadrilateral is the product of half the sums of the opposite sides; the exact area is the square root of the product of four sets of half the sum of the sides respectively diminished by sides.)

According to this verse the formula for the area of a cyclic quadrilateral is

$$\sqrt{S(S-a)(S-b)(S-c)(S-d)}$$

where *a*, *b*, *c*, *d* are four sides of a cyclic quadrilateral and $S=\dfrac{a+b+c+d}{2}$ and the one for the area of a triangle is

$$A=\sqrt{S(S-a)(S-b)(S-c)}$$

Brahmagupta's formula for the area of a cyclic quadrilateral $A=\sqrt{S(S-a)(S-b)(S-c)(S-d)}$ has been reproduced by Sridhar [1] (900), Mahāvira [2] (850) and Shripatit [3] (1039) : None of these writers has expressly mentioned the limitation that it holds only for an inscribed figure. Still it seems to have been implied by them. So this appears from the particular remark of Bhāskara II that the formula holds only in case of a special kind of quadrilaterals contemplated by them. Further, we find that examples of quadrilaterals viz. (4, 13, 14, 13), (25, 25, 39, 25) and (25, 39, 60, 52) given by Sridhar [4] and Prthūdakasvāmi [5] and those namely (14, 36, 61, 36), (169, 169, 407, 169) and (125, 195, 300, 260) given by Mahāvira [6] in illustration or the above formula, are all of the cyclic variety. Bhāskara II has shown that in the other cases, the above formula gives only an approximate value of the area of a quadrilateral.

For example,

भूमिश्चतुर्दशमिता मुखमङ्क कसङ्करूपं
बहू त्रयोदश – दिवाकर – सम्मितौ चेत्।
लम्बोऽपि यत्र रविसङ्ख्य एव तत्र
क्षेत्रे फलं कथय तत्कथितं तदाधैः।।170।।

(In a quadrilateral figure, of which the base is fourteen, the summit (mouth) nine, the flanks (sides) thirteen and twelve and the perpendicular twelve, tell the area as it was taught by the ancients.)

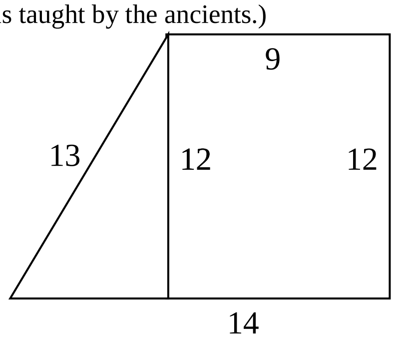

Fig. 1

Using the Brahmagupta's formula for the area of the quadrilateral, we find the area 419800 = 140.71 (approx) which is of the figure, the true area is 138.

The area of a triangle $A=\sqrt{S(S-a)(S-b)(S-c)(S-d)}$ given by Brahmagupta is found earlier in the Metrica of Heron of Alexandria. Therefore, the case for borrowing by Brahmagupta is very weak. In India, the result was derived for the quadrilateral first and then extended to cover the triangle or rather the triangle was stretched into a quadrilateral to bring it within the purview of the formula. Secondly and consequently, Brahmagupta's mode of derivation of the formula is entirely different from the Greek mode.

BRAHMAGUPTA'S THEOREM 2.

वर्णाश्रितभुजघातैक्यमुभयथान्योन्यभाजितं गुणयेत्।
योगेन भुजप्रतिभुजवधयोः कर्णौ पदे विषमे।।

(Br. Sp. Si XII. 28)

(The sums of the products of the sides about both the diagonals should be divided by each other and multiplied by the sum of the products of the opposite sides. The square roots of the quotients are the diagonals in visama quadrilateral.)

The second theorem of Brahmagupta gives the values of the diagonals of a cyclic quadrilateral. Now, this theorem is generally known as Brahmagupta's theorem and re-discovered [7] in Europe in 1619 AD by W. Snell.

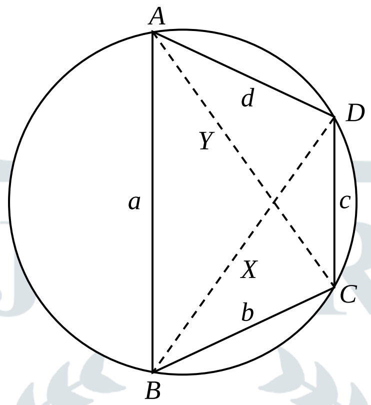

Fig. 2

If *a*, *b*, *c*, *d* are the lengths of the sides of a cyclic quadrilateral and *X* and *Y* are its diagonals then according to the theorem

$$X=\sqrt{\frac{(ab+cd)(ac+bd)}{(ab+bc)}} \qquad (1)$$

and

$$Y=\sqrt{\frac{(ad+bc)(ac+bd)}{(ab+cd)}} \qquad (2)$$

Thus enabling the diagonals to be calculated in terms of the sides.

From (1) and (2), we have

$$XY = ac + bd$$

i.e.

$$BD\cdot AC = AB\cdot CD + BC\cdot AD$$

which is known as Ptolemy's theorem. This shows that Ptolemy's theorem was first proved in India, therefore, the credit should go to the great Hindu Mathematician Brahmagupta for inventing the aforesaid values of *X* and *Y*.

SPEAKING ABOUT BRAHMAGUPTA'S GEOMETRY

F. Cajorig [8] has given his remark that Brahmagupta's theorem on cyclic quadrilateral is one of the remarkable achievement of Ancient Indian Mathematician.

Brahmagupta's treatment of the quadrilateral is limited to the cyclic quadrilateral. In fact, all his geometry is concerned with figures inscribable in circles, as a survey of his section on Kṣetragaṇita will convince anybody. [9] This fact was perhaps vaguely realised by Sridhara and Mahāvira and more clearly by Shripati, but Āryabhata II and Bhāskara II seem to have missed the significance of Brahmagupta's theorems completely. Hence we may conclude that Brahmagupta has not obtained the proof of the cyclic quadrilateral theorem but he has made new contribution in this theorem by giving the rules.

REFERENCES

[1] Tris, R. 43
[2] GSS, VII. 50
[3] Si Se, XIII. 28
[4] Tris, Ex 78, 79, 80
[5] vide his commentary on Br.sp.si, XII.21. Elsewhere (XII-26) he finds the circum-radii of these cyclic quadrilaterals.
[6] GSS, VII, 57, 58, 59 compare also VII $215\frac{1}{2}, 216\frac{1}{2}, 217\frac{1}{2},$ where it is required to find the diameters of the circles circumscribing these very.
[7] Smith, D.E. 1951. History of Mathematics, vol. II, Dover Publication, New York, Inc. p. 286.
[8] Cajori, F. 1919. A History of Mathematics, New York, p. 87.
[9] The aspect of Brahmagupta's geometry has been dealt with by the writer in a paper on "The cyclic quadrilateral in Indian Mathematics" presented at 21st Session of the All India Oriental Conference, 1961.