A NEW FIXED POINT RESULT FOR SINGLE VALUED MAPPING IN PARTIAL METRIC SPACES ENDOWED WITH A GRAPH

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Abstract : In this paper, in the setting of partial metric spaces, we present a new Suzuki type fixed point result for self- mapping satisfying an implicit relation. Our Result extends and generalizes the fixed point theorems in the existing literature.

Keywords: Partial metric space, fixed point, suzuki type contraction, implicit relation, graph

I. INTRODUCTION

Thousands of researchers have been attracted by fixed point theory since it occupies a prominent place in various field of mathematics. It provides very important tools for findings the existence and uniqueness of solutions. Banach in 1922, introduced a very important concept named as Banach contraction principle and it have become very popular due to iterations which can be easily implemented on the computers. Many of researchers generalized and extended Banach contraction principle in very different ways.

Matthews [10] introduced the notion of partial metric spaces, a generalization of metric space and proved most eminent fixed point result named as Banach Contraction Principle in the context of partial metric space. Notion of partial metric space have wide number of applications in the field of computer sciences such as computer domain and semanties. As a consequence, lot of authors have recently focused on partial metric spaces, see [1, 2, 8,13].
The study of implicit relation in partial metric spaces initiated by C. Vetro et al. [16] in 2013. Implicit relation on partial metric spaces have been used in many papers [4, 13, 14]. A lot of fixed point results and common fixed point results have been combined considering a general condition by an implicit relation in [11], [12] and in other papers. Proofs of some fixed point results becomes more simple using implicit relation.

Definition [2]: Let Q be any non-empty set. A partial metric on Q be a function p: $Q \times Q \rightarrow [0,\infty)$ satisfying the subsequent conditions:

(P_1) $p(\upsilon, \upsilon) = p(\upsilon, \tau) = p(\tau, \tau)$ if and only if $\upsilon = \tau$

(P_2) $p(\upsilon, \upsilon) \leq p(\upsilon, \tau)$

(P_3) $p(\upsilon, \tau) = p(\tau, \upsilon)$

(P_4) $p(\upsilon, \tau) \leq [p(\upsilon, \varkappa) + p(\varkappa, \tau)]- p(\varkappa, \varkappa)$

for all $\upsilon, \tau, \varkappa \in$ Q. Then the pair (Q, p) is known as partial metric space. A basic example of partial metric space: let Q = R^+ and p: $Q \times Q \rightarrow [0,\infty)$ be defined by

$$p(\upsilon, \tau) = \max\{\upsilon, \tau\} \text{ for all } \upsilon, \tau \in Q$$

Then (Q, p) is a partial metric space.

A very interesting approach in the theory of fixed point was recently given by Jachymski [9]. In recent investigation, the study of fixed point theory endowed with a graph plays an important role in many fields. Using this interesting idea, Jachymski studied the Banach contraction principle in metric spaces with a graph. His work extends and subsumes many recent results obtained on partially ordered metric spaces. Jachymski [9] uses the concept of graph instead of partial order in his paper. Graph theory have wide number of applications in many fields.

In this paper, let (Q, p, G) be complete partial metric space with the reflexive diagraph G= (V, E) where V(G) = X be the vertex set and E(G) be the edge set of G. Here, we take $\widetilde{G}$ be undirected graph obtained from G, $V(\widetilde{G})$=X and $E(\widetilde{G}) = E(G) \cup E(\widetilde{G})$. Motivated by the work of Ishak Altun et al. [3] in complete metric spaces for Suzuki type contraction, we prove a fixed point result in the setting of partial metric space concerning Suzuki type contraction and via implicit relation.

2. Implicit Relation

Implicit relations on metric spaces have been used in many papers, see [3, 7, 12, 13]. Let Φ be the set of all continuous functions J: $[0,\infty)^6 \to$ R which satisfies the following conditions:

J1: $J(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$ is non-increasing in variable $\sigma_2, \sigma_3, \ldots \sigma_6$

J2: there exist $\varsigma \in [0,1)$ such that

$$J(\sigma, \sigma', 0, (\sigma + \sigma'), \sigma, \sigma') \le 0$$

or $J(\sigma, \sigma', \sigma', \sigma, (\sigma + \sigma'), 0) \le 0$

or $J(\sigma, \sigma', \sigma', \sigma', \sigma', \sigma') \le 0$ for all $\sigma, \sigma' \ge 0$

implies

$$\sigma \le \varsigma \sigma'$$

J3: $J(\sigma', 0, 0, \sigma', \sigma', 0) > 0$ for all $\sigma' > 0$

Now we present some examples for J.

Examples 2.1. [3] Let J: $[0,\infty)^6 \to$ R be defined by

(i) $J(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - a(\sigma_3 + \sigma_4)$ where $a \in \left[0, \frac{1}{2}\right)$

(ii) $J(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - a \max\{\sigma_3, \sigma_4\}$ where $a \in \left[0, \frac{1}{2}\right)$.

(iii) $J(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - a \sigma_2$ where $a \in [0, 1)$.

(iv) $J(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - a(\sigma_5 + \sigma_6)$ where $a \in \left[0, \frac{1}{2}\right)$.

(v) $J(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = \sigma_1 - a \sigma_3 - b \sigma_4$ where $a, b \in \left[0, \frac{1}{2}\right)$.

3. Main Results:

Now inspired by [3], we state and prove the following theorems.

Theorem 3.1. Let (Q, p, G) be complete partial metric space with graph G and let $T: Q \to Q$ be a self-mapping. Define a nonincreasing function ψ from $[0,1)$ into $(\frac{1}{2}, 1]$ by

$$\psi(\varsigma) = \begin{cases} 1 & if \ 0 \le \varsigma \le \frac{1}{2} \\ 1 - \varsigma & if \ \frac{1}{2} \le \varsigma < 1 \end{cases}$$

Assume that there exist $\varsigma \in [0,1)$ such that

$$\psi(\varsigma) \, p(\upsilon, T\upsilon) \le p(\upsilon, \tau) \tag{1}$$

implies

$$J(p_b(T\upsilon, T\tau), p_b(\upsilon, \tau), p_b(\upsilon, T\upsilon), p_b(\tau, T\tau), p_b(\upsilon, T\tau), p_b(\tau, T\upsilon) - p_b(\tau, \tau)) \le 0 \tag{2}$$

for all $(\upsilon, \tau) \in E(\tilde{G})$ and for some $J \in \Phi$.

Also assume that the following conditions are satisfied:

1.   There exist υ_0 such that $(\upsilon_0, T\upsilon_0) \in E(\tilde{G})$ and T is edge preserving mapping that is $(\upsilon, \tau) \in E(\tilde{G})$ implies $(T\upsilon, T\tau) \in E(\tilde{G})$;

2.   for a sequence $\{\upsilon_n\}$ with $\upsilon_n = T\upsilon_{n-1}$ in Q converges to z and $(\upsilon_n, \upsilon_{n+1}) \in E(\tilde{G})$, we have $(\upsilon_n, \varkappa)$, $(z, T^n z) \in E(\tilde{G})$ $\forall$ n∈N and $\forall \varkappa \in$Q.

Then T has a fixed point.

Proof. There exist $\varkappa_0$ such that $(\varkappa_0, T\varkappa_0) \in E(\tilde{G})$ by assumption 1.

Since $\psi(\varsigma) \le 1$, so there exist $\psi(\varsigma) p(\varkappa_0, T\varkappa_0) \le p(\varkappa_0, T\varkappa_0)$ implies

$$J(p(T\varkappa_0, T^2\varkappa_0), p(\varkappa_0, T\varkappa_0), p(\varkappa_0, T\varkappa_0), p(T\varkappa_0, T^2\varkappa_0), p(\varkappa_0, T^2\varkappa_0), 0) \le 0$$

and from J1, we have

$$J(p(T\varkappa_0, T^2\varkappa_0), p(\varkappa_0, T\varkappa_0), p(\varkappa_0, T\varkappa_0), p(T\varkappa_0, T^2\varkappa_0), p(\varkappa_0, T\varkappa_0) + p(T\varkappa_0, T^2\varkappa_0), 0) \le 0$$

Therefore by J2, there exist $\varsigma \in [0,1)$ such that

$$p(T\varkappa_0, T^2\varkappa_0) \le \varsigma \, p(\varkappa_0, T\varkappa_0) \tag{3}$$

Again by 2, $(T\varkappa_0, T^2\varkappa_0) \in E(\tilde{G})$ and $\psi(\varsigma) p(T\varkappa_0, T^2\varkappa_0)) \le p(T\varkappa_0, T^2\varkappa_0))$ implies

$$J(p(T^2\varkappa_0, T^3\varkappa_0), p(T\varkappa_0, T^2\varkappa_0), p(T\varkappa_0, T^2\varkappa_0), p(T^2\varkappa_0, T^3\varkappa_0), p(T\varkappa_0, T^3\varkappa_0), 0) \le 0$$

Since J is non-increasing in variable σ_5 , therefore we have

$$J(p(T^2\varkappa_0, T^3\varkappa_0), p(T\varkappa_0, T^2\varkappa_0), p(T\varkappa_0, T^2\varkappa_0), p(T^2\varkappa_0, T^3\varkappa_0), p(T\varkappa_0, T^2\varkappa_0) + p(T^2\varkappa_0, T^3\varkappa_0), 0) \le 0$$

implies by J2, there exist $ş \in [0,1)$ such that
$$p(T^2 \varkappa_0, T^3 \varkappa_0) \leq ş\, p(T\varkappa_0, T^2 \varkappa_0) \tag{4}$$

Proceeding in this way, we obtain a sequence $\varkappa_n = T^n \varkappa = T\varkappa_{n-1}$ such that

$$
\begin{aligned}
p(\varkappa_n, \varkappa_{n+1}) &= p(T\varkappa_{n-1}, T\varkappa_n) \\
&= p(T\varkappa_{n-1}, T^2 \varkappa_{n-1}) \\
&\leq p(\varkappa_{n-1}, T\varkappa_{n-1} \leq \ldots \ldots \leq ş^n\, p(\varkappa_0, T\varkappa_0)
\end{aligned}
$$

which shows that the sequence $\{\varkappa_n\}$ is a Cauchy sequence, since $\sum_{n=1}^{n=\infty} p(\varkappa_n, \varkappa_{n+1}) < \infty$. As Q is a complete, so there exist z in Q such that $\varkappa_n$ converges to z and $\lim_{n \to \infty} p(\varkappa_n, z) = p_b(z, z) = 0$.

Next we will prove that
$$p(T\varkappa, z) \leq ş\, p(\varkappa, z) \quad \text{for all } \varkappa \in Q \text{ with } \varkappa \neq z. \tag{5}$$

For a $\varkappa \in Q$ with $\varkappa \neq z$, $\exists k_0 \in N$, such that $p(\varkappa_n, z) \leq p(\varkappa, z)/3$ for all $n \geq k_0$ then we have

$$
\begin{aligned}
\psi(ş)\, p(\varkappa_n, T\varkappa_n)) &\leq p(\varkappa_n, T\varkappa_n)) = p(\varkappa_n, \varkappa_{n+1}) \\
&\leq p(\varkappa_n, z) + p(z, \varkappa_{n+1}) \\
&\leq \tfrac{2}{3} p(\varkappa, z) \leq p(\varkappa, z) - \tfrac{p(x,z)}{3} \\
&\leq p(\varkappa, z) - p(\varkappa_n, z) \leq p(\varkappa_n, \varkappa) \tag{6}
\end{aligned}
$$

Hence by assumption 2 and by "Eq. 6", we have

$$J\left(\, p(T\varkappa_n, T\varkappa),\, p(\varkappa_n, \varkappa),\, p(\varkappa_n, T\varkappa_n),\, p(\varkappa, T\varkappa),\, p(\varkappa_n, T\varkappa),\, p(\varkappa, T\varkappa_n) - p(\varkappa, \varkappa)\,\right) \leq 0$$

or we can write
$$J(p(\varkappa_{n+1}, T\varkappa),\, p(\varkappa_n, \varkappa),\, p(\varkappa_n, \varkappa_{n+1}),\, p(\varkappa, T\varkappa),\, p(\varkappa_n, T\varkappa),\, p(\varkappa, \varkappa_{n+1})) \leq 0$$
since J increasing in variable $ϭ_6$.

Taking $n \to \infty$, it follows that
$$J(p(z, T\varkappa),\, p(z, \varkappa),\, 0,\, p(\varkappa, T\varkappa),)\, p(z, T\varkappa),)\, p(z, \varkappa)) \leq 0 \qquad \text{since } p(z, z) = 0$$

Since J is non-increasing in variable $ϭ_3$, so
$$J(p(z, T\varkappa),\, p(z, \varkappa),\, 0,\, p(\varkappa, z) + p(z, T\varkappa),\, p(z, T\varkappa),)\, p(z, \varkappa)) \leq 0$$

Hence by J2, we obtain
$$p(z, T\varkappa) \leq ş\, p(z, \varkappa) \quad \text{which proves "Eq.5"}$$

Next, we suppose that $T^k z \neq z \ \forall\, k \in N$, then from "Eq. 5", we get

$$p(T^{k+1} z, z) \leq ş^k\, p(Tz, z) \ \forall\, k \in N. \tag{7}$$

There are two case arises:

Case I: When $0 \leq ş \leq \tfrac{1}{2}$ then in this case, $\psi(ş) = 1$. Now, we show by induction that
$$p(T^n z, Tz) \leq ş\, p(z, Tz) \tag{8}$$
for $n \geq 2$. From (3), (8) holds for $n = 2$. Suppose that inequality (8) holds for some n with $n \geq 2$. Since
$$p(z, Tz) \leq p(z, T^n z) + p(T^n z, Tz) \leq p(z, T^n z) + ş\, p(z, Tz)$$
implies

$$p(z, Tz) \leq \tfrac{1}{1-ş} p(z, T^n z) \text{ and therefore}$$

$$
\begin{aligned}
\psi(ş)\, p(T^n z, T^{n+1} z) = p(T^n z, T^{n+1} z) &\leq ş^n\, p(z, Tz) \\
&\leq \tfrac{ş^n}{1-ş} p(z, T^n z) \leq \tfrac{ş^2}{1-ş} p(z, T^n z) \leq p(z, T^n z)
\end{aligned}
$$

That is
$$\psi(ş)\, p(T^n z, T^{n+1} z) \leq p(z, T^n z) \quad \text{and } (z, T^n z) \in E(G) \text{ by assumption 2, so it implies}$$

$$J(\, p(T^{n+1} z, Tz),\, p(T^n z, z),\, p(T^n z, T^{n+1} z),\, p(z, Tz),\, p(T^n z, Tz),\, p(z, T^{n+1} z) - p(z, z)\,) \leq 0$$

or $\quad J(\, p(T^{n+1} z, Tz),\, ş^{n-1} p(Tz, z),\, ş^n\, p(z, Tz),\, p(z, Tz),\, ş\, p(z, Tz),\, ş^n\, p(z, Tz)\,) \leq 0$

or $J(p(T^{n+1}z, Tz), p(T z, z), p(z, T z), p(z, Tz), p(z, Tz), p(z, T z)) \leq 0$

Therefore by J2, implies that

$$p(T^{n+1}z, Tz) \leq ş\, p(T z, z)$$
and thus "Eq.8" holds.

Next from "Eq.5", we obtain
$$p(T(T^n z), z) = p(T^{n+1}z, z) \leq ş\, p(T^n z, z) \leq ş^n\, p(T z, z)$$
implies that $T^n z \rightarrow z$ as $n \rightarrow \infty$ which contradict "Eq. 8".

Case II: When $\frac{1}{2} \leq ş < 1$ then in this case $\psi(ş) = 1 - ş$.
Next, we will prove that for all $n \geq 1$

$$(1\text{-}ş)\, p(\varkappa_{2n}, T\varkappa_{2n}) \leq p(\varkappa_{2n}, z)$$

or $(1\text{-}ş)\, p(\varkappa_{2n+1}, T\varkappa_{2n+1}) \leq p(\varkappa_{2n+1},\, z)$

Indeed, if
$$(1\text{-}ş)\, p(\varkappa_{2n}, T\varkappa_{2n}) >\, p(\varkappa_{2n}, z)$$

and $(1\text{-}ş)\, p(\varkappa_{2n+1}, T\varkappa_{2n+1}) >\, p(\varkappa_{2n+1},\, z)$
then we get

$$p(\varkappa_{2n}, T\varkappa_{2n}) \leq p(\varkappa_{2n}, z) + p(z, T\varkappa_{2n}) < (1\text{-}ş) [p(\varkappa_{2n}, T\varkappa_{2n}) + p(\varkappa_{2n+1}, T\varkappa_{2n+1})]$$
$$\leq (1\text{-}ş) [p(\varkappa_{2n}, T\varkappa_{2n}) + ş\, p(\varkappa_{2n}, T\varkappa_{2n})] = (1\text{-}ş^2)\, p(\varkappa_{2n}, T\varkappa_{2n})$$

That is $ş^2\, p(\varkappa_{2n}, T\varkappa_{2n}) \leq 0$ which is a contradiction.

Therefore, either
$$(1\text{-}ş)\, p(\varkappa_{2n}, T\varkappa_{2n}) \leq p(\varkappa_{2n}, z)$$
or $(1\text{-}ş)\, p(\varkappa_{2n+1}, T\varkappa_{2n+1}) \leq p(\varkappa_{2n+1},\, z)$ holds for all $n \in N$.

If $(1\text{-}ş)\, p(\varkappa_{2n}, T\varkappa_{2n}) = \psi(ş)\, p(\varkappa_{2n}, T\varkappa_{2n}) \leq p(\varkappa_{2n}, z)$ holds and also by assumption 2.,
$(\varkappa_{2n}, z) \in E(G)$ implies

$$J (p(T\varkappa_{2n}, Tz), p(\varkappa_{2n}, z), p(\varkappa_{2n}, T\varkappa_{2n}), p(z, Tz), p(\varkappa_{2n}, Tz), p(z, T\varkappa_{2n})\text{-}p(z,z)) \leq 0,$$

Or we can write latter inequality
$J (p(\varkappa_{2n+1}, Tz), p(\varkappa_{2n}, z), p(\varkappa_{2n}, \varkappa_{2n+1}), p(z, Tz), p(\varkappa_{2n}, Tz), p(z, \varkappa_{2n})) \leq 0$

Taking $n \rightarrow \infty$, we get
$$J(p(z, Tz), 0, 0, p(z, Tz), p(z, Tz), 0) \leq 0$$
which contradict J3.

If
$(1\text{-}ş)\, p(\varkappa_{2n+1}, T\varkappa_{2n+1}) \leq p(\varkappa_{2n+1},\, z)$ holds then similarly, we get a contradiction as we done above.

   Thus in all the cases, there exist $k \in N$, such that $T^k z = z$. Since $\{ T^k z \}$ is a cauchy sequence, we get
   $Tz = z$ i.e T has a fixed point z and from (5), uniqueness of fixed point can be easily proved

Corollary 3.2. Let (Q, p, G) be complete partial metric space with graph G and let $T: Q \rightarrow Q$ be a self-mapping. Define a nonincreasing function ψ from [0,1) into $(\frac{1}{2}, 1]$ by
$$\psi(ş) = \begin{cases} 1 & if\ 0 \leq ş \leq \frac{1}{2} \\ 1 - ş & if\ \frac{1}{2} \leq ş < 1 \end{cases}$$
Assume that there exist $ş \in [0,1)$ such that
   $\psi(ş)\, p(\upsilon, T\upsilon) \leq p(\upsilon, \tau)$ implies

   $p(T\upsilon, T\tau) \leq a\, p(\upsilon, T\upsilon) + b\, p(\tau, T\tau)$

for all $(\upsilon, \tau) \in E(\tilde{G})$, where a, b$\in [0, \frac{1}{2})$ and for some J$\in \Phi$.

Also assume that the following conditions are satisfied:
1. There exist υ_0 such that $(\upsilon_0, T\upsilon_0) \in E(\tilde{G})$ and T is edge preserving mapping that is $(\upsilon, \tau) \in E(\tilde{G})$ implies $(T\upsilon, T\tau) \in E(\tilde{G})$;
2. for a sequence $\{\upsilon_n\}$ with $\upsilon_n = T\upsilon_{n-1}$ in Q converges to z and $(\upsilon_n, \upsilon_{n+1}) \in E(\tilde{G})$, we have $(\upsilon_n, \varkappa),(z, T^n z) \in E(\tilde{G})$ $\forall$ n$\in$N and $\forall \varkappa \in$Q.

Then T has a fixed point.

Proof: We can deduce required result by taking J as in Example 2.1 (i) and Theorem 3.1

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