BETTI NUMBERS OF GRAPH IDEALS VIA COMBINATORIAL OF ALGEBRAIC TOPOLOGY

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Abstract : We apply some notions from combinatorial topology to establish various algebraic properties of graph ideals. In this way we provide new short proofs of some theorems from Betti numbers linearity, projective dimension as well as trees and forests. The intension is to find information about combinatorial objects, ie., graphs by studying the corresponding algebraic topology and vice versa. We use combinatorial topology to determine certain relations among the generating functions and use these to derive results regarding graded Betti numbers of graph edge ideals.

Key words: Betti numbers, edge ideal, projective dimension, forests, simplexial complex, minimal free resolution.

I. Introduction

In this paper we will study graphs in association with algebraic objects. Suppose G is a finite simple graph with vertex set [n] = {1, . . . , n} and edge set E(G), and let S := k[x_1 , . . . , x_n] denote the polynomial ring on n variables over some field k. We define the edge ideal $I_G \subseteq S$ to be the ideal generated by all monomials x_i x_j whenever ij ∈ E(G). The natural problem is to then obtain information regarding the algebraic invariants of the S-module $R_G := S/I_G$ in terms of the combinatorial data provided by the graph G. The study of edge ideals of graphs has become popular recently, an many papers have been written addressing various algebraic properties of edge ideals associated to various classes of graphs.

Graded Betti Numbers

Now we examine more closely Betti numbers by considering N-graded and N^n -graded Betti numbers[15]. We may also describe Betti numbers in terms of Tor. For an N^n -graded module M over R = k[x_1 , . . . , x_n] , tensoring a minimal free resolution of M,

$$0 \rightarrow R^{\beta_h} \rightarrow \cdots \rightarrow R^{\beta_1}$$

with k (considered as an R-module via k ≌ R/ x_1 , . . . , x_n) produces a complex

$$0 \rightarrow R^{\beta_h} \rightarrow \cdots \rightarrow R^{\beta_1}$$

where all the maps are zero. Taking ith homology of this complex we obtain

$$\beta_i(M) = \beta_i{}^k(M) = dim_k Tor_i{}^R(k, M)$$

For a (simple, finite) graph G we may write the minimal free resolution of k[Δ(G)] as

$$0 \rightarrow R^{\beta_h} \rightarrow \cdots \rightarrow R^{\beta_1} \rightarrow R^{\beta_0}$$

where R = R_k (G)

$$\beta_i(K[\Delta(G)] = dim_K(Tor_i{}^R(K[\Delta(G)], K))$$

and β_i = 0. Note that $\beta_0(K[\Delta(G)] = 1$

Observe that here we have pd_k (G) = pd(k[Δ(G)]) = h

Note also that $Tor_i{}^R(K[\Delta(G)], K)$ inherits the N^n - grading. For a ∈ N^n let $Tor_i{}^R(K[\Delta(G)], K)$ a denote the a graded component. This allows us to define N^n -graded Betti numbers and N-graded Betti numbers which will in some circumstances be easier to handle than the total Betti numbers.

II. Projective Dimension

The projective dimension of R_G given local information regarding the graph G. Recall that by Hochster's formula[12] the projective bound the projective dimension of R_G is the smallest integer such that

$$dim_K \overline{H}_{j-i-1}(Ind(\Delta[W])) = 0$$

for al $1 < i \leq j$ and subsets W of V (G) with j vertices. Hence if we know something about how the topological connectivity of Ind(G[W]) depends on the size of W we can bound the projective dimension

III. Generating functions of Betti numbers

The graded Betti numbers $\beta_{i,j}$ as coefficients of a certain generating function in two variables. We use combinatorial topology to determine certain relations among the generating function and use these to derive results regarding graded Betti numbers of edge ideals.

(i)We will say that two vertices in a graph are neighbours if and only if they are joined by an edge.

(ii) A graph in which no vertex is joined to itself by an edge is said to be simple.

(iii) A finite graph is one with finitely many vertices and finitely many edges.

(iv) The degree of a vertex is the number of edges to which it is joined.

(v) The degree of a graph is the maximum of the degrees of its vertices.

(vi) A terminal vertex is a vertex which is connected to at most one other vertex.

(vii)A subgraph G′ of the graph G is a graph such that V (G′) ⊆ V (G) and E(G′) ⊆ E(G).

(xi) If a ∈ V (G) then G \ {a} will denote the subgraph of G which has vertex set V (G) \ {a} and all the edges of G which do not feature a.

(xii) If e = {x, y} ∈ E(G) then G \ e will denote the graph with the same vertex set as G and edges E(G) \ e

(xiii) A tree is a connected graph with no cycles.

(xiv) A forest is a graph whose connected components are all trees.

Definition 1.1 : For a graph G on vertices $x_1 , . . . , x_n$ and a field k we define R_k (G) to be the polynomial ring over k in the in n indeterminants which we will also call $x_1 , . . . , x_n$, i.e.,

R_k (G) = k[$x_1 , . . . , x_n$], where k is any field, and we define I(G) to be the monomial ideal of R_k (G) generated by {$x_i x_j$ | {x_i , x_j } is an edge of G}. We call I(G) the graph ideal of G.

Definition 1.2: The projective dimension of a module M, pd(M), is the length of its minimal resolution (or ∞ if it has no finite resolution). We define the projective dimension of a graph G to be the projective dimension of the R_k (G)-module k[Δ(G)], and we will write

pd^k (G) = pd(k[Δ(G)]).

Definition 1.3: Let Γ be a simplicial complex with vertex set {$x_1 , . . . , x_n$ }.

Let R(Γ) = k[$x_1 , . . . , x_n$], for a field, k. The Stanley-Reisner[9] ideal of Γ, denoted by I(Γ), is the ideal of R(Γ) generated by all square free monomials $x_{i1} . . . x_{ij}$ such that there is no face of Γ with vertices $x_{i1} , . . . , x_{ij}$. The Stanley-Reisner Ring[9] (or face ring) of Γ is defined to be the quotient ring k[Γ] = R(Γ)/I(Γ).

Remark: Note that I(G) = I(Δ(G)), the Stanley-Reisner[9] ideal of the simplicial complex Δ(G) which has faces{$x_{i1} , . . . , x_{il}$ } |

no {x_{ij} , x_{ik} } is an edge of G . Consequently graph ideals are a special case of Stanley-Reisner ideals and we will henceforth write k[Δ(G)] for R_k (G)/I(G). We attempt to use the combinatorial properties of graphs to understand some of the algebraic properties of the associated ideals and vice versa.

Definition 1.4: A **free resolution** of an R- module M is a complex of free modules

F_0....... ,F_n ....

$$\mathfrak{I} : \dots\dots \xrightarrow{\varphi_{n-1}} F_n \xrightarrow{\varphi_n} \dots\dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

which is exact and is such that coker ϕ_1 = M . It is also a graded free resolution if R is a graded ring, the F_i are graded free modules and the maps are homogeneous of degree 0. A finite free resolution of length n is one in which $F_i = 0$ for all $i \geq n + 1$ but F_0 , . . . , F_n are all non zero. If

$$\mathfrak{I} : \dots\dots \xrightarrow{\varphi_{n-1}} F_n \xrightarrow{\varphi_n} \dots\dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

$$\mathfrak{S} : \dots\dots \xrightarrow{\psi_{n-1}} G_n \xrightarrow{\psi_n} \dots\dots \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

are free resolutions of M such that for all n there exists an isomorphism of modules, $\theta_n : F_n \to G_n$, and for all n we have $\theta_{n-1}\, \phi_n = \psi_n\, \theta_n$ then $\mathfrak{I}$ and $\mathfrak{S}$ will be said to be isomorphic resolutions.

Definition 1.5: A **minimal finite free resolution** of an R-module M is one with the smallest possible length and smallest possible rank for each of the free modules. Minimal resolutions are unique up to isomorphism. The rank of the ith free module in a minimal resolution is called the ith Betti number of M.

Definition 1.6 : For a ∈ N^n the ith **Betti number of degree** a of R/I

$$\beta_i{}^k(M) = dim_k Tor_i{}^R(k, M)$$

Definition 1.7 : The generating function of Betti numbers is

$$\mathcal{B}(G;x,y) = \sum_{i,j} \beta_{i,j}(G) x^{j-i} y^i$$

The two variables in $\mathcal{B}$(G;x,y) correspond to well known algebraic parameters of the edge ideal: the y-degree is the projective dimension of I_G and the x-degree is the regularity of I_G .

The generating function explicitly as

$$\boldsymbol{B}(G; x, y) = \sum_{i,j} \sum_{W \in \binom{V(G)}{j}} dim_k \tilde{H}_{j-i-1}(Ind(G[W]); k) x^{j-i} y^i$$

We wish to use to derive certain properties of edge ideals for some classes of graphs.

Example:

$\beta_{i,i+j}$	i=0	1	2
j=0	.	.	.
1	.	.	.
2	.	.	.
3	4	.	.
4	.	3	

5	.	3	3

Remark : If G is a graph with an isolated vertex v then $\mathcal{B}$(G;x,y) = $\mathcal{B}$(G\{v}:x,y)

Remark : (Hochster's Formula)[12]: Let k[Δ] = R(Δ)/I(Δ) be Stanley-Reisner ring of the Simplicial complex Δ. The non-zero Betti number of k[Δ] are only in squarefree degree b and may be expressed as $\beta_{i,b}\ (k[\Delta])\ =\ dim_k\widetilde{H}_{|b|-i-1}(\Delta_b;k)$

Hence the ith total Betti number may be expressed as

$$\beta_i\ (k[\Delta]) = \sum_{V\subseteq\{1,\dots n\}} dim_k\widetilde{H}_{|V|-i-1}(\Delta_V;k)$$

Theorem 1.8: If G is a graph with an isolated edge uv then

$\mathcal{B}$(G;x,y) = (1+xy) $\mathcal{B}$(G\{u,v}:x,y)

Proof:

For evey W⊆V(G) such that exactly one of {u,v} is in W we have that Ind(G[W]) is cone.

Hence

$dim_k\widetilde{H}_{j-i-1}(Ind(G[W]);k) = 0$. If {u,v}⊆ W⊆V(G)

Then Ind(G[W]) is a suspension of Ind(G[W\{u,v}])

We have,

$$dim_k\widetilde{H}_{j-i-1}(Ind(G[W]);k) = dim_k\widetilde{H}_{j-i-1}\big(susp(Ind(G[W]\{u,v\})):k\big)$$

$$= dim_k\widetilde{H}_{j-i-1}:k\ (Ind(G[W]\{u,v\});k)$$

$$= dim_k\widetilde{H}_{(j-2)-(i-1)}:k\ (Ind(G[W]\{u,v\});k)$$

In the definition of $\mathcal{B}$(G;x,y) involving Hochster's formula[12] we consider a sum over subsets W⊆V(G). We now split this sum according to the intersection {u,v} ∩W. If {u,v}∩W = ϕ

The partial sum is of course $\mathcal{B}$(G\{u,v}:x,y).

If exactly one of {u,v} is in W we have seen that the partial sum is 0.

If both {u,v} are in W then we use the formula

$$\sum_{i,j}\sum_{W\in\binom{V(G)}{j}} dim_k\widetilde{H}_{j-i-1}(Ind(G[W]);k)x^{j-i}y^i = \sum_{i,j}\sum_{u,v\in\binom{V(G)}{j}} \widetilde{H}_{(j-2)-(i-1)-1}\ (Ind(G[W]\{u,v\});k)x^{j-i}y^i$$

$$= xy\sum_{i,j}\sum_{u,v\in\binom{V(G)\{u,v\}}{j-2}} \widetilde{H}_{(j-2)-(i-1)-1}\ (Ind(G[W]\{u,v\});k)x^{(j-2)-(i-1)}y^{i-1}$$

= xy $\mathcal{B}$(G\{u,v}:x,y)

Theorem 1.9 : Let G be a graph with a vertex v and U a set of k vertices all different from v. If N(v) ⊆ N(u) for all u∈U, then for $\widetilde{U}\ \coloneqq U\cup\{v\}$ we have $\mathcal{B}$(G;x,y) = $\mathcal{B}$(G\{v}:x,y) + $(1+y)^k$ ($\mathcal{B}$(G\U; x,y) - $\mathcal{B}$(G\$\widetilde{U}$; x,y)

Proof:

$$\sum_{u,v\in\binom{V(G)}{j},|W\cap U|=l} dim_k\tilde{H}_{j-i-1}(Ind(G[W]);k) = \sum_{u,v\in\binom{V(G)}{j},|W\cap U|=l} dim_k\tilde{H}_{j-i-1}(Ind(G[W\backslash U]);k)$$

$$= \binom{k}{l}\sum_{u,v\in\binom{V(G)}{j},|W\cap U|=l} dim_k\tilde{H}_{j-i-1}(Ind(G[W\backslash U]);k)$$

$$= \binom{k}{l}(\beta_{i-1,j-1}(G\backslash U) - \beta_{i-1,j-1}(G\backslash\tilde{U}))$$

$$B(G;x,y) - B(G|\{v\};x,y) = \sum_{i,j}\sum_{W\in\binom{V(G)}{j}} dim_k\tilde{H}_{j-i-1}(Ind(G[W]);k)x^{j-i}y^i$$

$$= \sum_{i,j}\sum_{l=0}^{k}\sum_{v\in W\in\binom{V(G)}{j},|W\cap U|=l} dim_k\tilde{H}_{j-i-1}(Ind(G[W]);k)x^{j-i}y^i$$

$$= \sum_{i,j}\sum_{l=0}^{k}\binom{k}{l}(\beta_{i-1,j-1}(G\backslash U) - \beta_{i-1,j-1}(G\backslash\tilde{U}))x^{j-i}y^i$$

$$= \sum_{l=0}^{k}\binom{k}{l}y^l\sum_{i,j}(\beta_{i-1,j-1}(G\backslash U) - \beta_{i-1,j-1}(G\backslash\tilde{U}))\,x^{(j-l)-(i-l)}y^{i-l}$$

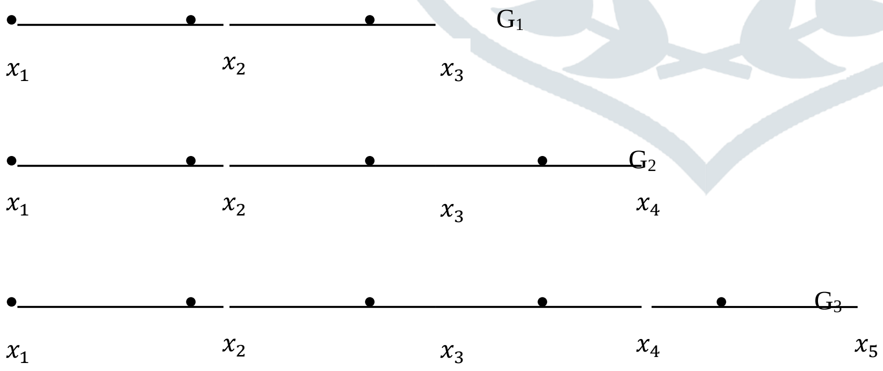
$$= \textstyle\sum_{l=0}^{k}\binom{k}{l}y^l \quad (\mathcal{B}(G\backslash U;x,y) - \mathcal{B}(G\backslash\tilde{U};x,y)$$

$$= (1+y)^k(\mathcal{B}(G\backslash U;x,y) - \mathcal{B}(G\backslash\tilde{U};x,y)$$

Remark : The projective dimension of the line graph is independent of the characteristic of the chosen field and is

$$pd(L_n) = \begin{cases} \frac{2n}{3} & if\ \ n \equiv 0\ mod\ 3 \\ \frac{2n-2}{3} & if\ n \equiv 1\ mod\ 3 \\ \frac{2n-1}{3} & if\ n \equiv 2\ mod\ 3 \end{cases}$$

Example: Let G_1,G_2, and G_3 be the following



These graphs are all trees and we shall see later that they therefore have projective dimension independent of the characteristic of our choice of field. The projective dimensions of $pd(G_1) = 2$, $pd(G_2) = 2$ and $pd(G_3) = 3$ So we have $pd(G_3 \backslash\{x_5\}) < pd(G_3)$ and $pd(G_2 \backslash \{x_4\}) = pd(G_2)$.

Proposition 1.10: Let G be a graph with an edge e = {a,b} such that b is a terminal vertex. Then $Pd^k(G \backslash \{b\}) \leq pd^k(G)$.

Proof:

First note that $V(G) \supseteq V(G \backslash \{b\}) = V(G) \backslash \{b\}$.

Let $p = pd(G \backslash \{b\})$.

We know that Hochster's Formula

$$\beta_p(G) = \sum_{W\subseteq V(G)} dim_k \breve{H}_{|W|-p-1}(\triangle_W; k)$$

$$\geq \sum_{W\subseteq V(G)\{b\}} dim_k \breve{H}_{|W|-p-1}(\triangle_W; k)$$

$$= \beta_p(G\{b\}) \neq 0$$

Therefore we may conclude that $pd^k(G) \geq pd^k(G \setminus \{b\})$.

Note: The projective dimension of T can be found from the projective dimensions of the subforests T' and T'' .

Theorem 1.11: The ith Betti number of degree d of the forest T may be expressed in terms of Betti numbers of the subgraphs T' and T'' as follows

$$\beta_{i,d}(T) = \beta_{i,d}(T') + \sum_{j=0}^{n-1}\binom{n-1}{j}\beta_{i-(j+1),\ d-(j+2)}(T'')$$

Proof:

We know that

$$\beta_{i,d}(T) = \beta_{i,d}(T') + \sum_{j=0}^{n-2}\binom{n-1}{j}\beta_{i-(j+1),\ d-(j+2)}(T'') + \sum_{j=0}^{n-2}\binom{n-1}{j}\beta_{i-(j+1),\ d-(j+3)}(T'')$$

We collect together the terms in Betti numbers of T'' to obtain

$$\sum_{j=0}^{n-2}\binom{n-1}{j}\beta_{i-(j+1),\ d-(j+2)}(T'') + \sum_{j=0}^{n-2}\binom{n-1}{j}\beta_{i-(j+1),\ d-(j+3)}(T'')$$

$$= \binom{n-2}{0}\beta_{i-1,d-2}(T'') + \left\{\binom{n-2}{1} + \binom{n-2}{0}\right\}\beta_{i-2,d-3}(T'') + \cdots$$

$$+ \left\{\binom{n-2}{n-2} + \binom{n-2}{n-3}\right\}\beta_{i-(n-1),d-n}(T'') + \binom{n-2}{n-2}\beta_{i-n,d-(n+1)}(T'')$$

$$= \sum_{j=1}^{n}\left\{\binom{n-2}{j-1} + \binom{n-2}{j-2}\right\}\beta_{i-j,d-(j+1)}(T'')$$

$$= \sum_{j=0}^{n-1}\binom{n-1}{j}\beta_{i-(j+1),d-(j+2)}(T'')$$

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