

# Closure Properties of some Bivariate Stochastic Ageing Classes

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## Abstract

In this paper, we determine the convolution of the bivariate ageing classes of some life time distributions.

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## 1. Introduction

In stochastic modeling, such life times are represented by random variables with respective distributions. Ageing properties play an important role in the analysis of life time distributions. Positive ageing signifies the averse effect on the age of the system. It may be caused by wearout or fatigue. A positive ageing class contains a life time distribution that show a characteristic behavior of positive ageing. Ageing properties are often expressed in terms of the residual life time of units at different ages. In particular, a probability distribution may belong to a certain positive ageing class, if the random residual life time of the corresponding unit decreases with increasing age. Thus, it is necessary to compare probability distributions.

The best studied univariate ageing class is the Increasing Failure Rate (IFR) class. A life time distribution with survival function  $\bar{F}$  belongs to this class, if the residual life time of a unit of less age dominates the residual life time of a unit of greater age with respect to stochastic order. Besides the IFR class, there is a variety of different positive ageing classes introduced in the literature. In particular, the closure properties of these classes with respect to the formation of coherent systems, convolutions and mixtures are considered. For further details, one may refer to Barlow and Proschan (1981). The IFR class is not closed under the formation of coherent systems with independent components. The smallest univariate class that contains the exponential distributions and that is closed under the formation of coherent systems with independent components and under limits in distribution is the Increasing Failure Rate Average (IFRA) class. The exponential distribution is the only distribution with a constant hazard rate. Therefore, the exponential distributions play a vital role in the study of ageing classes. Note here that the exponential distribution belongs to the IFR class and to its dual class, the Decreasing Failure Rate (DFR) class, because of its memoryless property. The DFR class is a negative ageing class.

The treatment of ageing concepts is predominantly restricted to univariate properties. Nevertheless, there were different approaches to find suitable bivariate versions of existing univariate classes.

In this paper, we focus on very strong versions of some Bivariate Ageing classes introduced and studied by Syed Tahir Hussainy (2015). We study the Closure of Some Bivariate Ageing Classes.

## 2. Preliminaries

In this section, we first give the definitions of some bivariate ageing classes, that are necessary for further discussion.

**Definition 2.1** The convolution of two bivariate functions  $f$  and  $g$  is given by

$$(f * g)(t, s) = \int_{-\infty}^{\infty} f(t - u, s - v) dg(u, v).$$

**Definition 2.2** Let  $\overline{F}_1$  and  $\overline{F}_2$  be two bivariate life time distributions. We say that  $\overline{F}_1$  is **stochastically smaller than**  $\overline{F}_2$  (written  $\overline{F}_1 \leq_{st} \overline{F}_2$ ), if  $\overline{F}_1(x, y) \leq \overline{F}_2(x, y)$ , for all  $x, y \geq 0$ .

**Definition 2.3** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate New Better Used than of order 2 (BNBU(2))**, if

$$\int_0^x \int_0^y \overline{F}(u + t, v + s) ds dt \leq \overline{F}(u, v) \int_0^x \int_0^y \overline{F}(t, s) ds dt$$

for  $u, v \geq 0$ .

**Definition 2.4** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate New Better than Used of order 3 (BNBU(3))**, if

$$\int_0^\infty \int_0^\infty \int_0^u \int_0^v \overline{F}(x + t, y + s) ds dt dv du \leq \overline{F}(x, y) \int_0^\infty \int_0^\infty \int_0^u \int_0^v \overline{F}(t, s) ds dt dv du,$$

for  $x, y \geq 0$ .

**Definition 2.5** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate New Better than Used in Average (BNBUA)**, if

$$\int_0^v \int_0^u \overline{F}(x + t, y + s) dt ds \leq \overline{F}(x, y) \int_0^v \int_0^u \overline{F}(t, s) dt ds,$$

for  $x, y \geq 0$ .

**Definition 2.6** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate New Better than Used in Convex Average (BNBUCA)**, if

$$\int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \overline{F}(x + t, y + s) ds dt dv du \leq \overline{F}(x, y) \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \overline{F}(t, s) ds dt dv du,$$

for  $x, y \geq 0$  and  $u, v > 0$ .

**Definition 2.7** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate New Renewal Better than Used (BNRBU)**, if

$$\overline{F}(x + t, y + s) \leq \overline{F}(x, y) \cdot \overline{W}(t, s)$$

for  $x, y, t, s \geq 0$ , where  $\overline{W}(t, s) = \int_s^\infty \int_t^\infty \overline{F}(u, v) dv du$ .

**Definition 2.8** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate New Better than used in Average at Specific Interval (BNBUASI)**, if

$$\int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \overline{F}(t + u, s + v) du dv \leq \overline{F}(t, s) \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \overline{F}(u, v) dv du$$

for  $t, s \geq 0$  and  $x_2, y_2 \geq 0, y_1, x_1 \geq 0$ ,

**Definition 2.9** A bivariate random variable  $(X, Y)$  or its distribution  $\overline{F}(x, y)$  is said to have **Bivariate Exponential Better than Used (BEBU)**, if

$$\overline{F}(x + t, y + s) \leq \overline{F}(t, s) \cdot \exp \left[ - \left( \frac{x+y}{\mu} \right) \right],$$

for  $x, y, t, s \geq 0$ , where  $\mu = \int_0^\infty \int_0^\infty \overline{F}(x, y) dy dx$ .

**Theorem 2.1** Suppose that  $\overline{F}_1$  and  $\overline{F}_2$  are two independent BNBU(2) life time distributions. Then their convolution is a BNBU(2).

**Proof.** The survival function of the convolution of two life time distributions  $\overline{F}_1$  and  $\overline{F}_2$  is

$$\overline{F}(t_1, t_2) = \int_0^\infty \overline{F}_1(t_1 - u, t_2 - v) d\overline{F}_2(u, v).$$

Consider

$$\begin{aligned} & \int_0^x \int_0^y \overline{F}(x + t, y + s) ds dt \\ &= \int_0^x \int_0^y \left[ \int_0^\infty \overline{F}_1(x + t - w, y + s - z) d\overline{F}_2(w, z) \right] ds dt \\ &= \int_0^\infty \left[ \int_0^x \int_0^y \overline{F}_1(x + t - w, y + s - z) ds dt \right] d\overline{F}_2(w, z) \\ &\leq \int_0^\infty \left[ \overline{F}_1(x - w, y - z) \int_0^x \int_0^y \overline{F}_1(t, s) ds dt \right] d\overline{F}_2(w, z) \\ &= \left\{ \int_0^\infty \overline{F}_1(x - w, y - z) d\overline{F}_2(w, z) \right\} \left\{ \int_0^x \int_0^y \overline{F}_1(t, s) ds dt \right\} \\ &\leq \overline{F}(x, y) \int_0^x \int_0^y \overline{F}(t, s) ds dt, \end{aligned}$$

for all  $x, y \geq 0$ . The inequality in the third step follows from the fact that  $\overline{F}_1$  is BNBU(2) and the last inequality is because  $\overline{F}_1 \leq_{st} \overline{F}$ . This completes the proof.

**Theorem 2.2** Suppose that  $\overline{F}_1$  and  $\overline{F}_2$  are two independent BNBU(3) life distributions then their convolution

is BNBU(3).

**Proof.** The survival function of the convolution of two life time distributions  $\overline{F}_1$  and  $\overline{F}_2$  is

$$\overline{F}(t_1, t_2) = \int_0^\infty \overline{F}_1(t_1 - u, t_2 - v) d\overline{F}_2(u, v).$$

Consider

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^u \int_0^v \overline{F}(x + t, y + s) dt ds dv du \\ &= \int_0^\infty \int_0^\infty \int_0^u \int_0^v \left[ \int_0^\infty \overline{F}_1(x + t - z, y + s - w) d\overline{F}_2(z, w) \right] dt ds dv du \\ &= \int_0^\infty \left[ \int_0^\infty \int_0^\infty \int_0^u \int_0^v \overline{F}_1(x + t - z, y + s - w) ds dt dv du \right] d\overline{F}_2(z, w) \\ &\leq \int_0^\infty \left[ \overline{F}_1(x, y) \int_0^\infty \int_0^\infty \int_0^u \int_0^v \overline{F}_1(t - z, s - w) ds dt dv du \right] d\overline{F}_2(z, w) \\ &= \overline{F}_1(x, y) \int_0^\infty \int_0^\infty \int_0^u \int_0^v \left[ \int_0^\infty \overline{F}_1(t - z, s - w) d\overline{F}_2(z, w) \right] ds dt dv du \\ &\leq \overline{F}_1(x, y) \int_0^\infty \int_0^\infty \int_0^u \int_0^v \overline{F}(t, s) ds dt dv du \end{aligned}$$

The first inequality is because  $\overline{F}_1$  is BNBU(3) and the last inequality is because  $\overline{F}_1 \leq \overline{F}$ . It follows that  $\overline{F}$  is BNBU(3). This completes the proof.

**Theorem 2.3** Suppose that  $\overline{F}_1$  and  $\overline{F}_2$  are two independent BNBUA life time distributions. Then their convolution is a BNBUA.

**Proof.** The survival function of the convolution of two life time distributions  $\overline{F}_1$  and  $\overline{F}_2$  is

$$\overline{F}(t_1, t_2) = \int_0^\infty \overline{F}_1(t_1 - u, t_2 - v) d\overline{F}_2(u, v).$$

Consider

$$\begin{aligned} & \int_0^v \int_0^u \overline{F}(x + t, y + s) ds dt \\ &= \int_0^v \int_0^u \left[ \int_0^\infty \overline{F}_1(x + t - w, y + s - z) d\overline{F}_2(w, z) \right] dt ds \\ &= \int_0^\infty \left[ \int_0^v \int_0^u \overline{F}_1(x + t - w, y + s - z) dt ds \right] d\overline{F}_2(w, z) \\ &\leq \int_0^\infty \left[ \overline{F}_1(x, y) \int_0^v \int_0^u \overline{F}_1(t - w, s - z) dt ds \right] d\overline{F}_2(w, z) \\ &= \overline{F}_1(x, y) \int_0^v \int_0^u \left[ \int_0^\infty \overline{F}_1(t - w, s - z) d\overline{F}_2(w, z) \right] dt ds \\ &= \overline{F}_1(x, y) \int_0^v \int_0^u \overline{F}(t, s) dt ds \\ &\leq \overline{F}(x, y) \int_0^v \int_0^u \overline{F}(t, s) dt ds \end{aligned}$$

The first inequality is because  $\overline{F}_1$  is BNBUA and the inequality in the last step is because  $\overline{F}_1 \leq_{st} \overline{F}$ . This completes the proof.

**Theorem 2.4** Suppose that  $\overline{F}_1$  and  $\overline{F}_2$  are two independent BNBUCA life time distributions. Then their convolution is a BNBUCA.

**Proof.** The survival function of the convolution of two life time distributions  $\overline{F}_1$  and  $\overline{F}_2$  is

$$\overline{F}(t_1, t_2) = \int_0^\infty \overline{F}_1(t_1 - u, t_2 - v) d\overline{F}_2(u, v).$$

Consider

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \bar{F}(x+t, y+s) dt ds dv du \\ &= \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \left[ \int_0^\infty \bar{F}_1(x+t-w, y+s-z) d\bar{F}_2(w, z) \right] dt ds dv du \\ &= \int_0^\infty \left[ \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \bar{F}_1(x+t-w, y+s-z) dt ds dv du \right] d\bar{F}_2(w, z) \\ &\leq \int_0^\infty \bar{F}_1(x-w, y-z) \left[ \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \bar{F}_1(t, s) dt ds dv du \right] d\bar{F}_2(w, z) \\ &= \left[ \int_0^\infty \bar{F}_1(x-w, y-z) d\bar{F}_2(w, z) \right] \left[ \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \bar{F}_1(t, s) dt ds dv du \right] \\ &\leq \bar{F}(x, y) \int_0^\infty \int_0^\infty \int_u^\infty \int_v^\infty \bar{F}(t, s) dt ds dv du \end{aligned}$$

The first inequality is because  $\bar{F}_1$  is BNBUCA and the inequality the last step is because  $\bar{F}_1 \leq \bar{F}$ . It follows that  $\bar{F}$  is BNBUCA. This completes the proof.

**Theorem 2.5** Suppose that  $\bar{F}_1$  and  $\bar{F}_2$  are two independent BNRBU life time distributions. Then their convolution is BNRBU.

**Proof.** The survival function of the convolution of two life time distributions  $\bar{F}_1$  and  $\bar{F}_2$  is 
$$\bar{F}(t_1, t_2) = \int_0^\infty \bar{F}_1(t_1 - u, t_2 - v) d\bar{F}_2(u, v).$$

Consider

$$\begin{aligned} \bar{F}(x+t, y+s) &= \int_0^\infty \bar{F}_1(x+t-w, y+s-z) d\bar{F}_2(w, z) \\ &\leq \int_0^\infty \left[ \bar{F}_1(x-w, y-z) \int_s^\infty \int_t^\infty \bar{F}_1(u, v) dv du \right] d\bar{F}_2(w, z) \\ &= \int_s^\infty \int_t^\infty \bar{F}_1(u, v) dv du \int_0^\infty \bar{F}_1(x-w, y-z) d\bar{F}_2(w, z) \\ &= \bar{F}_1(x, y) \int_s^\infty \int_t^\infty \bar{F}(u, v) dv du \\ &\leq \bar{F}(x, y) \int_s^\infty \int_t^\infty \bar{F}(u, v) dv du \end{aligned}$$

The first inequality is because  $\bar{F}_1$  is BNRBU and the inequality in the last step is because  $\bar{F}_1 \leq_{st} \bar{F}$ . This completes the proof.

**Theorem 2.6** Suppose that  $\bar{F}_1$  and  $\bar{F}_2$  are two independent BNBUASI life time distributions. Then their convolution is a BNBUASI.

**Proof.** The survival function of the convolution of two life time distribution  $\bar{F}_1$  and  $\bar{F}_2$  is 
$$\bar{F}(t_1, t_2) = \int_0^\infty \bar{F}_1(t_1 - u, t_2 - v) d\bar{F}_2(u, v).$$

Consider

$$\begin{aligned} & \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \bar{F}(t+u, s+v) du dv \\ &= \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \left[ \int_0^\infty \bar{F}_1(t+u-w, s+v-z) d\bar{F}_2(u, v) \right] du dv \\ &= \int_0^\infty \left[ \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \bar{F}_1(t+u-w, s+v-z) du dv \right] d\bar{F}_2(u, v) \\ &\leq \int_0^\infty \left[ \bar{F}_1(t, s) \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \bar{F}_1(u-w, v-z) du dv \right] d\bar{F}_2(u, v) \\ &= \bar{F}_1(t, s) \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \left[ \int_0^\infty \bar{F}_1(u-w, v-z) d\bar{F}_2(u, v) \right] du dv \\ &= \bar{F}_1(t, s) \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \bar{F}(u, v) du dv \\ &\leq \bar{F}_1(t, s) \int_{y_1}^{y_1+y_2} \int_{x_1}^{x_1+x_2} \bar{F}(u, v) du dv \end{aligned}$$

The First equality is because  $\bar{F}_1$  is BNBUASI and the inequality in the last step is because  $\bar{F}_1 \leq_{st} \bar{F}$ . This completes the proof.

**Theorem 2.7** Suppose that  $\bar{F}_1$  and  $\bar{F}_2$  are two independent BEBU life distributions then their convolution is a BEBU.

**Proof.** The survival function of the convolution of two life distribution  $\bar{F}_1$  and  $\bar{F}_2$  is

$$\bar{F}(t_1, t_2) = \int_0^\infty \bar{F}_1(t_1 - u, t_2 - v) d\bar{F}_2(u, v).$$

Consider

$$\begin{aligned} \bar{F}(x + t, y + s) &= \int_0^\infty \bar{F}_1(x + t - u, y + s - v) d\bar{F}_2(u, v) \\ &= \int_0^\infty \bar{F}_1[x + (t - u), y + (s - v)] d\bar{F}_2(u, v) \\ &\leq \int_0^\infty \exp\left(-\frac{x+y}{\mu}\right) \bar{F}_1(t - u, s - v) d\bar{F}_2(u, v) \\ &\leq \exp\left(-\frac{x+y}{\mu}\right) \bar{F}_1(t, s) \end{aligned}$$

The first inequality is because  $\bar{F}_1$  is BEBU and the last inequality is because  $\bar{F}_1 \leq_{st} \bar{F}$ . This completes the proof.

### 3. Conclusion

In this paper, we derived the convolution of bivariate ageing classes BNBU(2), BNBU(3), BNBUA, BNBUCA, BNRBU, BNBUASI, BEBU of life time distributions.

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