STRONG BI-IDEALS IN C_1 AND C_2 NEAR SUBTRACTION SEMIGROUPS

P.ANNAMALAI SELVI[1]

[1]Research Scholar

Dr.S.JAYALAKSHMI[2]

[2]Head and Associate Professor

Department of Mathematics, Sri Parasakthi College for Women, Courtallam, Tamilnadu.

Affiliated to ManonmaniamSundaranar University, Abishekapatti, Tirunelveli, Tamilnadu, India.

Abstract:

In this paper we introduce the notation of strong bi-ideals in C_1 and C_2 near-subtraction semigroup and study some of their properties.

Key words:

Near subtraction semigroups, X.S.I, GNF, strong bi-ideals in C_1 and C_2 near-subtraction semigroup.

1. Introduction

B.M.Schein [8] considered systems of the form (X; o;/), where X is a set of functions closed under the composition "o" of functions (and hence (X; o) is a function semigroup) and the set theoretic subtraction "/" (and hence (X;/) is a subtraction algebra in the sense of [1]). Y.B.Jun et al[5] introduced the notation of ideals in subtraction algebras and discussed the characterization of ideals. For basic definition one may refer to Pilz[7]. Mahalakshmi et al. [6] studied the notation of bi-ideals in near subtraction semigroups. Annamalai Selvi et al. [2] studied the notation of bi-ideals in C_1 and C_2 - near subtraction semigroups. The purpose of this paper is to introduce the notation of strong bi-ideals in C_1 and C_2near-subtraction semigroups. We investigate some basic results, examples and properties.

2.Preliminaries

Definition:2.1. A nonempty set X together with binary operations **'' – ''** is said to be **subtraction algebra** if it satisfies the following conditions

(i) $x - (y - x) = x$.

(ii) $x - (x - y) = y - (y - x)$.

(iii) $(x - y) - z = (x - z) - y$, for every $x, y, z \in X$.

Definition:2.2. A nonempty set X together with two binary operations **''–''** and **''•''** is said to be a **subtraction semigroup** if it satisfies the following conditions

(i) $(X, -)$ is a subtraction algebra.

(ii) $(X, \bullet)$ is a semigroup.

(iii) $X(y - z) = xy - xz$ and $(x-y)z = xz - yz$, for every $x, y, z \in X$.

Definition:2.3. A non empty set X together with two binary operations '' – ''and' '•' 'is said to be a **right near subtraction semigroup** if it satisfies the following conditions

(i) (X, −) is a subtraction algebra.
(ii) (X, •) is a semigroup.
(iii) (x − y) z = xz – yz, for every x, y, z ∈ X.

It is clear that 0x = 0, for all x∈X. Similarly we can define a left near- subtraction semigroup. Here after a near–subtraction semigroup means only a right near-subtraction semigroup.

Definition:2.4. A nonempty subset S of a subtraction semigroup X is said to be a s**ubalgebra** of X, if x – y ∈ S, for all x, y ∈ S.

Definition:2.5. Let (X, - , .) be a near – subtraction semigroup. A nonempty subset I of X is called

(i)A **left ideal** if I is a subalgebraof(X , –) and xi – x (y – i) ∈ I for all x , y ∈ Xand i∈ I .

(ii) A **right ideal** I is a subalgebraof(X , –) and IX ⊆ I.

(iii) If I is both a left and right ideal then, it is called a **two-sided ideal** (simply, ideal) of X.

Definition:2.6. A near subtraction semigroup X is said to be **Zero – symmetric** if x0 = 0 for every $x \in$ X.

Definition:2.7. An element e ∈ X is said to be **idempoten**t if for each e ∈ X, $e^2 =$ e.

Definition:2.8. A subalgebra Q of (X , −) is said to be a **quasi-ideal** of zero-symmetric near subtraction semigroup of X if QX∩XQ ⊆ Q.

Definition: 2.9. A subalgebra B of (X , −) is said to be a **bi-ideal** of zero-symmetric near subtraction semigroup of X if BXB⊆ B.

Definition:2.10. A bi-ideal B of (X , −) is said to be a **strong bi-ideal** of X if $XB^2 \subseteq B$.

Definition:2.11. We say that X is an **s (*s′*) near subtraction semigroup** if a ∈ Xa (aX),

for all a∈X.

Definition:2.12. A near subtraction semigroup X is said to be **sub commutative** if aX = Xa, for every a∈X.

Definition:2.13. A near subtraction semigroup X is said to be **left bi-potent** if Xa = X a^2, for every a∈X.

Definition:2.14. An element a∈X is said to be **regular** if for each a∈X, a = aba, for some b∈X.

Definition:2.15. An element a∈X is said to be **strongly regular** if for each a∈X, a = b a^2, for some b∈X.

Definition:2.16. A non empty subset of X is called

(i) a **left X-subalgebra** of X is a subalgebra of (X , –) and XA ⊆ A.

(ii) a **right X-subalgebra** of X is a subalgebra of (X , –) and AX ⊆ A.

(iii) an invariant X- subalgebra of X if A is both left and right X-subalgebras of X.

Definition:2.17. A near subtraction semigroup X is called an **X.S.I (X- subalgebra invariance)** near subtraction semigroup if every left X – subalgebra of X is also a right X- subalgebra.

Definition:2.18. A near subtraction semigroup X is called a **generalized near-field (GNF)** if for each a∈X there exists a unique b∈X such that a = aba and b = bab.

Definition:2.19. X is said to be a near subtraction semigroup of **left permutable (Type I)** if

(ab) c = (ba) c, for all a, b, c ∈ X.

Definition:2.20. X is said to be a near subtraction semigroup of **right permutable (Type II)** if

a (bc) = a (cb), for all a, b, c ∈ X.

3. Strong Bi-ideals in C_1 and C_2 near-subtraction semigroup

In this section we define strong bi-ideals in C_1 and C_2 near-subtraction semigroups and give some examples of these new concepts.

Definition:3.1. Let X be a right near subtraction semigroup. If for all x∈ X, xX = xXx then we say X is a C_1-near subtraction semigroup.

Definition:3.2. Let X be a right near subtraction semigroup. If for all x∈ X, Xx = xXx then we say X is a C_2- near subtraction semigroup.

Example:3.3. Let X ={0,a,b,c} be the Klein's four group.Define subtraction and multiplication in X as follows:

–	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Here (X, –, .) is a near subtraction semigroup (see [[7], pg.408] scheme15 (0, 13, 0, 13)).

Clearly { 0, b } is a strong bi-ideal of X.

Remark:3.4. Every strong bi-ideal is bi-ideal. But the converse not true.

Example 3.5. Let X ={0, 1, 2} in which subtraction and multiplication are defined by

-	0	1	2
0	0	0	0
1	1	0	2
2	2	0	0

.	0	1	2
0	0	0	0
1	0	0	1
2	0	0	2

Here (X, –, .) is a near subtraction semigroup (see [[7], pg.407] scheme 2(0, 0, 1)).

Clearly { 0, 2} is bi-ideal. But not strong bi-ideal, since X $\{0, 2\}^2$ = X ⊄ { 0, 2 }.

Theorem:3.6. Let X be a s, C_1 - near subtraction semigroup. If X is strongly regular if and only if B $=XB^2$, for every strong bi-ideal B of X.

Proof: Let B be a strong bi-ideal of X and let b ∈ B.

Since X is strongly regular, there exists x ∈ X such that b = $Xb^2 \in XB^2$, for some b ∈ B.

(i.e) B ⊆ XB^2. From the definition of a strong bi-ideal XB^2 ⊆ B.

Therefore B = XB^2, for every strong bi-ideal B of X.

Conversely, let b ∈ X and Xb is a strong bi-ideal of X.

Let $b \in Xb = X\ (Xb)^2 = XXbXb = XbXb = XbXbb\ \in Xb^2$. Therefore b ∈ X$b^2$.

Hence X is strongly regular.

Theorem:3.7. Let X be a s, C_1- near subtraction semigroup then X is left bi-potent if and only if B = XB^2, for every strong bi-ideal B of X.

Proof: Every left bi-potent is strongly regular and by Theorem:3.6, the result is true.

Theorem:3.8. Let X be a s, C_1- near subtraction semigroup then B = BXB, for every

bi-ideal B of X if and only if B = XB^2.

Proof: Assume that B = BXB, for every bi-ideal B of X.

Let $x \in B = BXB\ then\ x = bx_1b\ \in bXb = bXbb \in XXB^2\ \subseteq XB^2$,for some$b \in B\ and\ x_1 \in X$. By the definition of strong bi-ideal $XB^2 \subseteq B$. Therefore B = XB^2.

Conversely, assume that B = XB^2 , for every strong bi-ideal B of X.

By Remark: 3.14 [2], X is strongly regular.

By Theorem: 3.15 [2], $hence\ B = BXB,$ for every bi-ideal B of X.

Proposition:3.9. Let X be a C_1 - near subtraction semigroup and B be a strong bi-ideal of X then B is a quasi-ideal of X.

Proof: Let $X \in BX \cap XB\ then\ x = b_1 n_1 = n_2 b_2, for\ some\ b_1, b_2 \in B\ and\ n_1, n_2 \in X$

Since X be a C_1- near subtraction semigroup. By remark 3.14[2], X is strongly regular,

$b_1 = c\, {b_1}^2\ and\ b_2 = d\, {b_2}^2, for\ some\ c, d \in X.$

$Hence\ x = b_1 n_1 = (c\, {b_1}^2) n_1 = cb_1 b_1 n_1 = cb_1 n_2 b_2 = cb_1 n_2 (d\, {b_2}^2) \in XB^2 \subseteq B.$

Therefore $x \in B \Longrightarrow BX \cap XB \subseteq B,$ hence B is a quasi-ideal of X.

Theorem:3.10. Let X be a C_1 - near subtraction semigroup and B is a bi-ideal of X. If B is a strong bi-ideal of X if and only if B is a quasi-ideal of X.

Proof: Only if part follows from Proposition:3.9.

Conversely, assume that B is a quasi-ideal of X then B is a bi-ideal of X.

If $x \in XB^2\ then\ x = n\, b^2 = nbb \in Xb \Longrightarrow x \in XB.$

Since X is a C_1 - near subtraction semigroup, $x \in bX = bXb \in BX \Rightarrow x \in BX.$

Therefore $x \in BX \cap XB \subseteq B \Longrightarrow x \in B. Therefore\ XB^2 \subseteq B.$

Hence B is a strong bi-ideal of X.

Theorem:3.11. Let X be a C_2 be a near subtraction semigroup, then B is a bi-ideal of X if and only if B is a strong bi-ideal of X.

Proof: If part is trival.

Suppose B is a bi-ideal of X.Let $x \in XB^2\ then\ x = n\, b^2 = nbb \in Xb = bXb \in BXB \subseteq B$

Since X be a C_2 be a near subtraction semigroup. Therefore $XB^2 \subseteq B.$

Hence B is a strong bi-ideal of X.

Theorem:3.12. Let X be a s- right permutable C_1- near subtraction semigroup if and only if X is strongly regular.

Proof: Assume that X is strongly regular. Therfore X is regular, for every x ∈ X, x = xax

Let y ∈ xX then y = xx_1 = xaxx_1= xax_1x ∈ xXXx ⊆ xXx.(where x_1∈ X and X is right permutable). Therfore y ∈ xXx ⟹ xX ⊆ xXx………(1)

Obviously xXx ⊆ xX …………..(2)

From (1) and (2), we get xX = xXx (i.e) X is a C_1- near subtraction semigroup.

Conversely, assume that X is a C_1- near subtraction semigroup (i.e) xX = xXx.

Let x ∈ xX = xXx = xXxx = xXx^2 ∈ XXx^2 ⊆ Xx^2. Therefore x ∈ Xx^2.

Hence X is strongly regular.

Theorem:3.13. Let X be a s- left permutable C_2- near subtraction semigroup if and only if X is strongly regular.

Proof: Assume that X is strongly regular. Therfore X is regular, for every x ∈ X, x = xax.

Let y ∈ Xx then y = x_1 x = $x_1$xax= xx_1ax ∈ xXXx ⊆ xXx.(where x_1∈ X and X is left permutable). Therfore y ∈ xXx ⟹ Xx ⊆ xXx………(1)

Obviously xXx ⊆ Xx …………..(2)

From (1) and (2), we get xX = xXx (i.e) X is a C_2- near subtraction semigroup.

Conversely, assume that X is a C_2- near subtraction semigroup (i.e) Xx = xXx.

Let x ∈ Xx = xXx = xxXx . Then x = xxx_1x =x x_1x^2 ∈ XXx^2 ⊆ Xx^2. Therefore x ∈ Xx^2.

Hence X is strongly regular.

Theorem:3.14. Let X be a left permutable s- near subtraction semigroup then the following are equivalent.

(i) $XB^2 = B$, for every strong bi-ideal of X.

(ii) B = BXB for every bi-ideal B of X.

(iii) X is regular and X is a X.S.I near subtraction semigroup.

(iv) X is C_1-near subtraction semigroup and for all left X-subalgebra M_1 *and* M_2 *of* X, $M_1 \cap M_2 = M_1\ M_2$.

(v) Xx ∩ Xy = Xxy, for all x, y ∈ X.

(vi) X is left bi-potent.

(vii) A = $\sqrt{A}$, for every left X-subalgebra A of X.

(viii) X is strongly regular.

(ix) aXa = Xa = Xa^2 for all a ∈ X.

(x) X is a C_2near subtraction semigroup.

(xi) X is regular and X is GNF.

Proof: **(i) ⇒ (ii)** Assume that $XB^2 = B$, for every strong bi-ideal of X.

By Proposition: 5.1.13[6], hence B= BXB, for every bi-ideal B of X.

(ii)⇒ (iii) Assume that B = BXB for every bi-ideal B of X.

Let Xa is a bi-ideal of X.

Let a ∈ Xa = XaXXa ⊆ XaXa , then a = x_1ax_2a = $x_1x_2\ a^2$ ∈ XXa^2 ⊆ Xa^2 ⟹ a ∈ Xa^2.

Therefore a = $x_1a^2 = x_1aa = ax_1a \in aXa$ (Since X is a left permutable)

Therefore X is regular.

If A is any left X-subalgebra of X, then by Proposition:414[6], A is an ideal and thus A is an invariant X-subalgebra of X. Therefore X is a X.S.I – near subtraction semigroup.

(iii) ⇒ (iv) Since X is regular. Therefore X is a C_1-near subtraction semigroup.

let x ∈ $M_1 \cap M_2$, for some x∈ M_1 and x ∈ M_2.

Let x ∈ xX = xXx [since X is a C_1 -near subtraction semigroup]

$= xXxx \in M_1$XX$M_2 \subseteq M_1M_2$

Therefore $M_1 \cap M_2 \subseteq M_1M_2$ ……….(1)

Let x∈ M_1M_2 then x = yz, for some y∈ M_1 and z ∈ M_2.

Now x = yz ∈ yX = yXy ∈ X $M_1 \subseteq M_1$ [since X is a C_1 -near subtraction semigroup]

Also x = yz ∈ X $M_2 \subseteq M_2$ [since X is a left X- subalgebra]

Therefore $M_1M_2 \subseteq M_1 \cap M_2$ ………(2)

From (1) and (2), hence $M_1 \cap M_2$= M_1M_2.

(iv) ⇒ (v) Let $M_1 = Xx\ and\ M_2 = Xy, for\ some\ x, y \in X.$

By assumption Xx∩Xy = XxXy. Now Xx = Xx∩X = XxX ⟹ Xxy = XxXy.

Therefore Xx∩Xy = Xxy.

(v) ⇒ (vi) Since X is a s- near subtraction semigroup.

Let a ∈ Xa = Xa∩Xa = Xaa = X a^2 ⟹ Xa ⊆ X a^2. Obviously X a^2 ⊆ Xa.

Hence X is left bi-potent.

(vi) ⇒ (vii) Clearly A ⊆ $\sqrt{A}$. Let a ∈ $\sqrt{A}$. Then a^n ∈ A, for some positive integer n.

Also Xa = X a^2 = X a^n is a left bi-potent - near subtraction semigroup.

Since X is a s- near subtraction semigroup. Let a ∈ Xa = Xa^n ⟹ a = ba^n∈ XA ⊆ A and hence a ∈ A, $\sqrt{A}$ ⊆ A. Therefore A = $\sqrt{A}$, for every left X-subalgebra A of X.

(vii) ⇒ (viii) Let 0 ≠ a ∈ X. Now a^3 ∈ Xa^2 so that a ∈ $\sqrt[3]{Xa^2}$ = Xa^2.

Therefore X is strongly regular.

(viii) ⇒ (ix) Let X be strongly regular.Then Xa = Xa^2, X is a left permutable s- near subtraction semigroup, Xa^2 = aXa. Therefore Xa = Xa^2 = aXa.

(ix) ⇒ (x) Assume that aXa = Xa = Xa^2 for all a ∈ X.

Therefore X is a C_2near subtraction semigroup, Xa = aXa.

(x) ⇒ (xi) Since X is a C_2near subtraction semigroup, a ∈ Xa = aXa, for a ∈ X.

Therefore X is regular.

Let a ∈ Xa = aXa then a = ax_1a = x_1a^2 ∈ Xa^2. Therefore X is strongly regular.

Then by Lemma:3.17[2], eXe = eX, for every e ∈ E. Since X is C_2- near subtraction semigroup. Therefore Xe = eXe and so eX = eXe = Xe for every e ∈ E.

Again by the Lemma: 3.18 [2], E ⊆ C(X). Hence X is GNF.

(xi) ⇒ (i) Assume that X is GNF.

Then X is regular and sub commutative near subtraction semigroup.

Therefore $XB^2 = B$, for every strong bi-ideal of X.

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