Application of Convexity in Applied Mathematics.

Tarun Kumar Dey

Assistant Professor, Department of Mathematics, Sambhu Nath College, Labpur, Birbhum, West Bengal, India. Email: deytarunkumar2020@gmail.com

Abstract

Convex functions are closely related to convex sets. Convex functions have many important properties that are very useful in optimization. The uses of convex technologies and homothetic production functions have an important role in optimum resource allocation. The convexity property may be used for the adoption of eco-friendly mechanism to restore the balance between flora and fauna. The present paper analyses the application of convexity property in the theory of production regarding the optimum allocation of resources in a broader perspective.

Keywords: convexity; duality; homothetic; dynamic

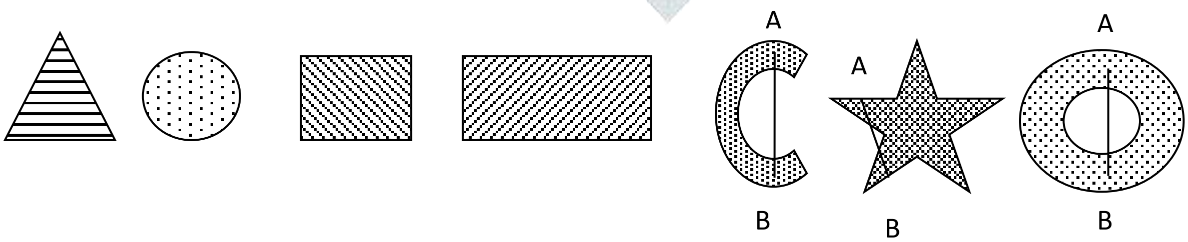
1. Convex Set

A set is called convex iff any straight line (joining by two points from the set) entirely lie on the set. Suppose x∈A, y∈A, then the joining straight line,

$\overline{xy}$ = tx + (1-t) y∈A, t∈ [0, 1]

Now tx + (1-t) y is called the convex combination. When t=0, y∈A and t=1 x∈A. An intermediate values of t i.e. 0< $t < 1$ gives us the weighted average of x & y.

The solid triangle, circle, square, rectangle etc are the examples of convex set because any straight line drawn from taking any two points lies entirely on the set. A crescent shape or a star, or a hollow circle is not convex set. Straight line like $\overline{AB}$ does not lie entirely on the set.



2. Properties

Any vector is an extreme point of a convex set if it cannot be expressed as a convex combination of two other vectors in the set i.e. an extreme point does not lie on the line segment between any other two vectors in the set.

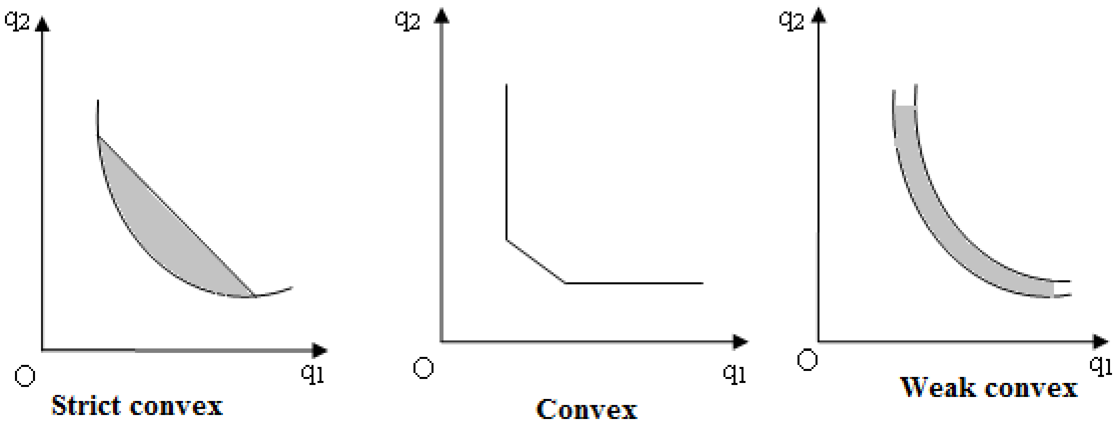
Any vector is a closed and bounded convex set with a finite number of extreme points can be expressed as a convex combination of the extreme point.

The solution space of a set of simultaneous linear equations is a convex set having a finite number of extreme points (R Bronson and G Naadimuthu 1997).

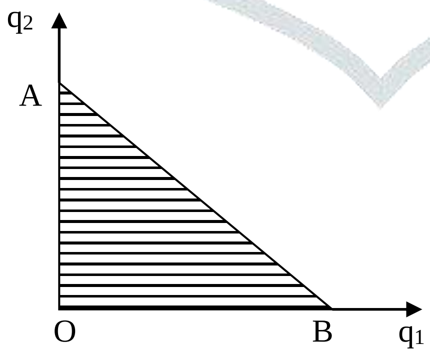
Without loss of generality here we assume the non-negativity restriction. Then a set X can be set can be defined as

$X = \{x = (x_1, x_2, x_3, x_4, \ldots, xn), x_i \geq 0, \forall i = 1,2,3, \ldots \ldots, n. \}$

Now the set X is a closed, convex set iff $tx_1 + (1-t)\, x_2$, $\forall t$, $0<t<1$



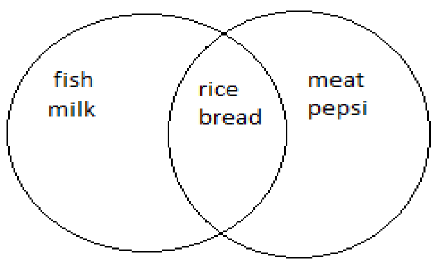
△ OAB is a closed convex set. Any point on this set is affordable. At corner point O there is zero solution. The other two extreme points i.e. at A & B one of X is chosen.



Suppose A & B are two convex sets. Then A intersection B is a convex set but A union B may or may not B a convex sets.

A = {rice, bread, fish, milk}

B = {rice, bread, meat, pepsi}



3. Convex function

Convex functions are closely related to convex sets. If f(x) is convex, then for any constant, say k, it can give rise to a convex set, say S (AC Chiang 1984).

If S ≡ $\{x|f(x) \le K\}$ then f(x) is convex.

If S ≡ $\{x|f(x) \ge K\}$ then f(x) is concave.

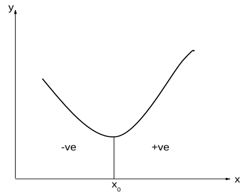
Suppose Y = f(x) be differentiable and second order derivative exist.

Then f(x) will be strictly convex iff f $f''(x) > 0 \rightarrow$ f(x) has a minimum value.

Then f(x) will be strictly concave iff f $f''(x) < 0 \rightarrow$ f(x) has a maximum value.

Suppose Y = f(x) be differentiable and strictly convex and it has a minimum at point x = x_0.

Then $f'(x) < 0$ for $x < x_0$



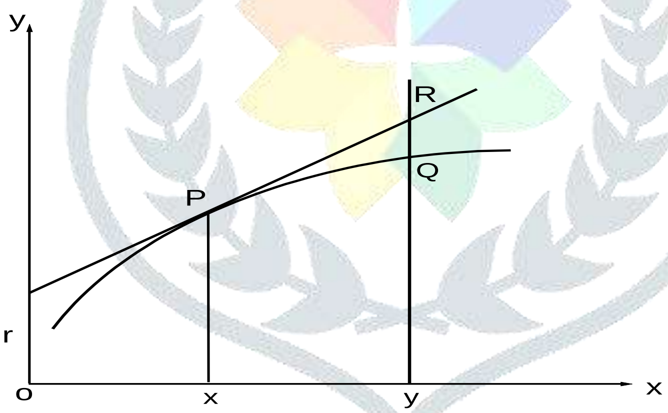
$f'(x) = 0$ for $x = x_0$

$f'(x) > 0$ for $x > x_0$

$f'(x)$ Changes sign from –ve to +ve at point x_0

The situation will be reversed if the function is concave.

Let y = f(x) is concave iff any tangent drawn at point P lie entirely above or on the graph (C Birchenhall and P Grout 1987).



t(y) = r + $f'(x)$y. At the tangency point f(x) = t(x) and x = y

f(x) = r + $f'(x)x \rightarrow r = f(x) - f'(x)x$

$\therefore t(y) = f(x) - f'(x)x + f'(x)y$

Or $t(y) = f(x) + f'(x)(y - x)$

Now if Q lies below R then $f(y) \le t(y) = f(x) + f'(x)(y - x)$

Thus y = f(x) is concave iff $f(y) \le f(x) + f'(x)(y - x)$

And y = f(x) is convex iff $f(y) \ge f(x) + f'(x)(y - x)$

4. Hessian-Determinant to check convexity:

Let multivariate function, z=f(x, y).

Then f(x, y) will be convex iff $z_{xx}>0$ and $z_{xx}.z_y > (z_{xy})^2$.

And f(x, y) will be concave iff $z_{xx}<0$ and $z_{xx}.z_y > (z_{xy})^2$.

A convenient or sufficient test or second order condition for convexity/concavity is the Hessian Determinant.

$$|H| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix}$$

When $|H_1| > o \text{ and } |H_2| > 0$, the Hessian is called +ve definite and the function are convex and the function has a minimum value. When $|H_1| < o \text{ and } |H_2| > 0$, the Hessian is called -ve definite and the function are concave and the function has a maximum value (E T Dowling 1986).

The idea of convex sets and convex/concave functions has many important properties that are very useful techniques applied in mathematics particularly in various optimizing problems. Suppose f(x) is concave then - f(x) will be convex i.e. one is the mirror image of the other. Thus minimization of one will maximize the other and hence it is used in case of duality. As a result if we analyse the properties and applications of convexity the other is automatically done. To find global/local maximum/minimum and in duality the idea of convexity is very much helpful.

5. Convexity property under uncertain behaviour

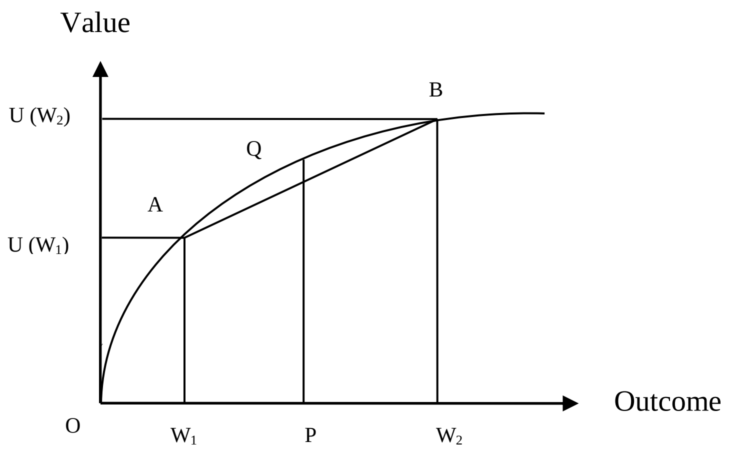
For the sake of simplicity here we assume that function can be measured the return on gambles on monitory terms, is strictly increasing and is continuous with first and second order derivatives exist. In this situation the expected outcome of a gamble:
$E[W] = pW_1 + (1-p) W_2$ where $W_i, \forall i = 1,2$ are different outcome levels and p is the probability of outcome of lottery (0<p<1) (J Henderson and R E Quandt 1987).

Now the person is risk neutral if the expected value of the lottery equals the expected of value of the lottery i.e.

$U(pW_1 + (1-p) W_2) = p\,U(W_1) + (1-p)\,U(W_2)$

In this situation expected function will be 45^0 line i.e. he/she has no preference either in gambling or certainty.

Now if the person is risk averse then the expected function will be concave. If we take any two points A and B on the expected function then the chord must lie above the joining line $\overline{AB}$ (as shown in the following figure).



$$U (pW_1 + (1-p) W_2) > p\, U (W_1) + (1-p)\, U (W_2)$$

Thus we can say that concave function assured risk aversion. In this situation the person acceptance set A (W) must be convex A (W) be the set of all gambles the he would accept at an initial level of W.

Now in case of risk lover or plunger the value is less than its expected i.e.

$$U (pW_1 + (1-p) W_2) < p\, U (W_1) + (1-p)\, U (W_2)$$

Arrow-Pratt used the following formula to measure the absolute risk aversion:

$$r(W) = -\frac{r''(W)}{r'(W)}$$

Now if r (W) <0 implies the function is strictly concave and the person is risk averter. And if r (W) >0 implies the utility function is strictly convex and the person is risk lover.

6. Minkowski addition of sets

Let $Q_1 = [0, 1]^2$, $Q_2 = [1, 2]^2$, $Q_3 = [1, 3]^2$

Then $Q_3^2 = Q_1^2 + Q_2^2$

From Minkowski addition of sets we get the sum of the squares Q_1^2 and Q_2^2 is the square of $Q_1 + Q_2$ i.e. $Q_1^2 + Q_2^2 = (Q_1 + Q_2)^2$

Suppose a firm has the following production process:

	Lab. Cap.	Lab. Cap.	Lab. Cap.
Q_1	[0, 1]	[1, 2]	[2, 3]
Q_2	[1, 2]	[2, 3]	[3, 4]
Q_3	[1, 3]	[3, 5]	[5, 7]

Activity Q_3 is derived from Q_1 and Q_2 i.e. simply adding Q_1 and Q_2.

In production process $Q^2_{L+K} = Q_L^2 + Q_K^2$

Though the cost of productions in $Q_1 + Q_2$ is the same in Q_3 it is better to use the activity Q_3 to get the advantage of large scale production process. Here Q_1 may be called small-scale production process, Q_2

may be called medium-scale production process Q_3 may be called large-scale production process. By means of the combination of Q_1 and Q_2, Q_3 will be a possible and efficient method using increasing returns to scale.

7. Conclusion

The global temperature is increasing day by day and the amount of GHGs in the atmosphere also rising. As a consequence the fertility of land has been reduced due to over exploitation, excessive use of chemical fertilisers, insecticides and pesticides. Due to indiscriminate deforestation the amount of rainfall reduces and land erosion takes place. Natural calamities like droughts, floods, cyclones, global warming, melting glaciers, raising sea level etc. are increasing and environment is degraded. Finally it breaks the eco-system and distorts the balance between flora and fauna. The adoption of suitable convex techniques we can at least minimize it at considerable level.

References

[1] Birchenhall C, Grout P (1987) Mathematics for Modern Economics. Heritage Publisher, New Delhi, p 132-152

[2] Bronson R, Naadimuthu G (1997) Theory and Problems of Operation Research. 2nd edn. Schaum's Outline Series, McGraw-Hill, Singapore, p 21-22

[3] Cairns RD (2007): Exhaustible Resources; Non-convexity and Competitive Equilibrium, Environment Resource Econ (2008) 40: 177-193, Accessed on 31st July 2007, Springer Science + Business Media B.V.

[5] Danilov VI, Koshevoy GA (2006): Discrete Convexity, Journal of Mathematical Science Vol. 133, No. 4, Accessed on June 2007, Science + Business Media

[6] Dowling ET (1986) Theory and Problems of Mathematics for Economists. International edn. Schaum's Outline Series, McGraw-Hill, Singapore, p138-139