

# Properties and Hierarchy of Upper and Lower $z$ -Perfectly Continuous Multifunctions

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**Abstract.** Basic properties of upper and lower  $z$ -perfectly multifunctions are studied and their place in hierarchy of variants of continuity of multifunctions, that already exists in the literature is discussed. The distinctiveness of upper (lower)  $z$ -perfectly continuous multifunction from that of other variants of continuity of multifunctions in the mathematical literature is studied.

**Keywords:** strongly continuous multifunctions, upper (lower) perfectly continuous multifunctions, upper (lower)  $z$ -supercontinuous multifunctions, upper (lower)  $z$ -continuous multifunctions, upper (lower) semicontinuous multifunctions, upper (lower) cl-supercontinuous multifunctions, upper (lower) almost cl-supercontinuous multifunctions, upper (lower) quasi  $z$ -supercontinuous

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## 1 Introduction

T. Noiri [32] introduced the concept of perfectly continuous function and basic properties of perfectly continuous functions are further elaborated in [20]. The class of strongly continuous function defined by Levine [27] is strictly contained in the class of cl-supercontinuous ( $\equiv$  clopen continuous) functions as defined by Reilly and Vamanamurthy [35]. Taking account of [23], we can mention some remarkable classes of function in descending order, each class properly contains the next class  $z$ -perfectly continuous function [24], pseudo perfectly continuous function [23], quasi-perfectly continuous functions [25],  $\delta$ -perfectly continuous [19] strongly continuous functions of Levine [27].

In this paper we extend the notions and results of variants of continuity of functions to the realm of multifunctions (see for example [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [15], [26], [29], [30], [31], [32], [35], [36], [37], [38]).

In the present paper the notion of upper and lower  $z$ -perfectly continuous multifunctions [30] and elaborate upon their place in the hierarchy of variants of continuity of multifunctions that already exist in the literature.

Section 2 is devoted to the preliminaries and basic definitions. In section 3 we give definition the notion of upper and lower  $z$ -perfectly continuous multifunctions and discuss the interrelations that exists

among them and other variants of continuity of multifunction. In Section 4 we study new properties of upper  $z$ -perfectly continuous multifunctions. In section 5, we study lower  $z$ -perfectly continuous multifunctions and give their characteristics when the space is Alexandroff.

## 2 Preliminaries and Basic Definitions

**Definition 2.1 ([16])** A multifunction  $\varphi: X \rightarrow Y$  from a topological space  $X$  into a topological space  $Y$  is said to be

- (1) strongly continuous if  $\varphi^{-1}(B)$  is clopen in  $X$  for every subset  $B \subset Y$ .
- (2) upper perfectly continuous if  $\varphi^{-1}(V)$  is clopen in  $X$  for every open set  $V \subset Y$ .
- (3) lower perfectly continuous if  $\varphi_+^{-1}(V)$  is clopen in  $X$  for every open set  $V \subset Y$ .
- (4) upper completely continuous if  $\varphi^{-1}(V)$  is regular open in  $X$  for every open set  $V \subset Y$ .
- (5) lower completely continuous if  $\varphi_+^{-1}(V)$  is regular open in  $X$  for every open set  $V \subset Y$ .

**Definition 2.2** A multifunction  $\varphi: X \rightarrow Y$  from a topological space  $X$  into a topological space  $Y$  is said to be

- (1) upper  $z$ -supercontinuous [4] if for each  $x \in X$  and each open set  $V$  containing  $\varphi(x)$ , there exists a cozero set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (2) lower  $z$ -supercontinuous [4] if for each  $x \in X$  and each open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a cozero set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (3) upper  $D_\delta$ -supercontinuous [15] if for each  $x \in X$  and each open set  $V$  containing  $\varphi(x)$ , there exists a regular  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (4) lower  $D_\delta$ -supercontinuous [5] if for each  $x \in X$  and each open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a regular  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (5) upper  $D$ -supercontinuous [2] if for each  $x \in X$  and each open set  $V$  containing  $\varphi(x)$ , there exists an open  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (6) lower  $D$ -supercontinuous [2] if for each  $x \in X$  and each open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists an open  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (7) upper  $D^*$ -supercontinuous [12] if for each  $x \in X$  and each open set  $V$  containing  $\varphi(x)$ , there exists a strongly open  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (8) lower  $D^*$ -supercontinuous [2] if for each  $x \in X$  and each open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a strongly open  $F_\sigma$ -set  $U$  containing  $x$  such that such  $\varphi(z) \cap V \neq \phi$ , for each  $z \in U$ .

- (9) upper strongly  $\theta$ -continuous [26] if for each  $x \in X$  and each open set  $V$  containing  $\varphi(x)$ , there exists a  $\theta$ -open set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (10) lower strongly  $\theta$ -continuous [26] if for each  $x \in X$  and each open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a  $\theta$ -set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .

**Definition 2.3** A multifunction  $\varphi: X \rightarrow Y$  from a topological space  $X$  into a topological space  $Y$  is said to be

- (a) upper (lower) perfectly continuous (respectively almost perfectly continuous, respectively quasi perfectly continuous, respectively  $\delta$ -perfectly continuous) [12] if  $\varphi^{-1}(U)$  ( $\varphi_+^{-1}(U)$ ) is clopen in  $X$  for every open (respectively regular open, respectively  $\theta$ -open, respectively  $\delta$ -open) subset  $U$  of  $Y$ .
- (b) upper quasi  $z$ -supercontinuous ( $D_\delta$ -supercontinuous) ([15], [21]) if for each  $x \in X$  and each  $\theta$ -open set  $V$  containing  $\varphi(x)$ , there exists a cozero set (regular  $F_\sigma$ -set)  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (c) lower quasi  $z$ -supercontinuous ( $D_\delta$ -supercontinuous) ([15], [21]) if for each  $x \in X$  and each  $\theta$ -open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a cozero set (regular  $F_\sigma$ -set)  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (d) upper (lower) quasi  $\theta$ -continuous (faintly continuous) [17] if for every  $\theta$ -open set  $V \subset Y$ ,  $\varphi^{-1}(V)$  ( $\varphi_+^{-1}(V)$ ) is  $\theta$ -open (open) in  $X$ .

**Definition 2.4** A multifunction  $\varphi: X \rightarrow Y$  from a topological space  $X$  into a topological space  $Y$  is said to be [(a)]

- (a) upper (almost) cl-supercontinuous ([11], [13], [18]) (respectively  $z$ -supercontinuous ([4], [22]), respectively  $D_\delta$ -supercontinuous [5], respectively strongly  $\theta$ -continuous [26]) at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \subset V$ , there exists a clopen set (respectively cozero set, respectively regular  $F_\sigma$ -set, respectively  $\theta$ -open set)  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (b) lower (almost) cl-supercontinuous ([11], [13], [18]) (respectively  $z$ -supercontinuous ([4], [22]), respectively  $D_\delta$ -supercontinuous [5], respectively strongly  $\theta$ -continuous [26]) at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a clopen set (respectively cozero set, respectively regular  $F_\sigma$ -set, respectively  $\theta$ -open set)  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$ ; for each  $z \in U$ .
- (c) upper supercontinuous ( $\delta$ -continuous) [1] if for each  $x \in X$  and each open (regular open) set  $V$  containing  $\varphi(x)$ , there exists a regular open set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .

- (d) lower supercontinuous ( $\delta$ -continuous) [1], if for each  $x \in X$  and each open (regular open) set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a regular open set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$ , for each  $z \in U$ .
- (e) upper quasi cl-supercontinuous [17] if for each  $x \in X$  and each  $\theta$ -open set  $V$  containing  $\varphi(x)$  there exists a clopen set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (f) lower quasi cl-supercontinuous [17] if for each  $x \in X$  and each  $\theta$ -open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a clopen set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$ ; for each  $z \in U$ .

**Definition 2.5** A multifunction  $\varphi: X \rightarrow Y$  from a topological space  $X$  into a topological space  $Y$  is said to be

- (i) upper (almost) completely continuous [14] if  $\varphi^{-1}(V)$  is regular open in  $X$  for every (regular) open set  $V \subset Y$ .
- (ii) lower (almost) completely continuous [14] if  $\varphi_+^{-1}(V)$  is regular open in  $X$  for every (regular) open set  $V \subset Y$ .
- (iii) upper (almost) perfectly continuous [14] if  $\varphi^{-1}(V)$  is clopen in  $X$  for every (regular) open subset  $V$  of  $Y$ .
- (iv) lower (almost) perfectly continuous [14] if  $\varphi_+^{-1}(V)$  is clopen in  $X$  for every (regular) open subset  $V$  of  $Y$ .
- (v) upper (almost)  $z$ -supercontinuous ([4], [15]) at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \subset V$ , there exists a cozero set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (vi) lower (almost)  $z$ -supercontinuous ([4], [15]) at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a cozero set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  each  $z \in U$ .
- (vii) upper (almost)  $D_\delta$ -supercontinuous ([5], [15]) at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \subset V$ , there exists a regular  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (viii) lower (almost)  $D_\delta$ -supercontinuous ([5], [15]) at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \cap V \neq \phi$ ; there exists a regular  $F_\sigma$ -set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (ix) upper (almost) strongly  $\theta$ -continuous [26] at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \subset V$ , there exists a  $\theta$ -open set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (x) lower (almost) strongly  $\theta$ -continuous [26] at  $x \in X$  if for each open (regular open) set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a  $\theta$ -open set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .

- (xi) upper supercontinuous ( $\delta$ -continuous) [3] if for each  $x \in X$  and each open (regular open) set  $V$  containing  $\varphi(x)$ , there exists a regular open set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (xii) lower supercontinuous ( $\delta$ -continuous) [3] if for each  $x \in X$  and each open (regular open) set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a regular open set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (xiii) upper cl-supercontinuous [11] at  $x \in X$  if for each open set  $V$  with  $\varphi(x) \subset V$ , there exists a clopen set  $U$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (xiv) lower cl-supercontinuous [11] at  $x \in X$  if for each open set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a clopen set  $U$  containing  $x$  such that  $\varphi(z) \cap V \neq \phi$  for each  $z \in U$ .
- (xv) upper almost cl-supercontinuous [13] if for each  $x \in X$  and each regular open set  $V$  in  $Y$  containing  $\varphi(x)$  there exists a clopen set  $U$  in  $X$  containing  $x$  such that  $\varphi(U) \subset V$ .
- (xvi) lower almost cl-supercontinuous [13] if for each  $x \in X$  and each regular open set  $V$  in  $Y$  with  $\varphi(x) \cap V \neq \phi$ , there exists a clopen set  $U$  in  $X$  containing  $x$  such that  $\varphi(x) \cap V \neq \phi$  for each  $x \in U$ .

### 3 Upper and lower $z$ -perfectly continuous

**Definition 3.1 ([30])** A multifunction  $\varphi: X \rightarrow Y$  from a topological space  $X$  into a topological space  $Y$  is

- (a) [30] upper  $z$ -perfectly continuous at  $x \in X$  if for each cozero set  $V$  with  $\varphi(x) \subset V$ ,  $\exists$  a clopen set  $U$  in  $X$  containing  $x$  such that  $\varphi(U) \subset V$ . The multifunction is said to be upper  $z$ -perfectly continuous if it is upper  $z$ -perfectly continuous at each  $x \in X$ .
- (b) [30] lower  $z$ -perfectly continuous at  $x \in X$  if for each cozero set  $V$  with  $\varphi(x) \cap V \neq \phi$ , there exists a clopen set  $U$  in  $X$  containing  $x$  such that  $\varphi(x) \cap V \neq \phi$  for each  $x \in U$ . The multifunction is said to be lower  $z$ -perfectly continuous if it is lower  $z$ -perfectly continuous at each  $x \in X$ .

The interrelations among variations of continuity of multifunctions defined in Definitions 2.1, 2.2, 2.3, 2.4, 2.5 and upper (lower)  $z$ -perfectly continuity of multifunctions is illustrated in Figure 1.

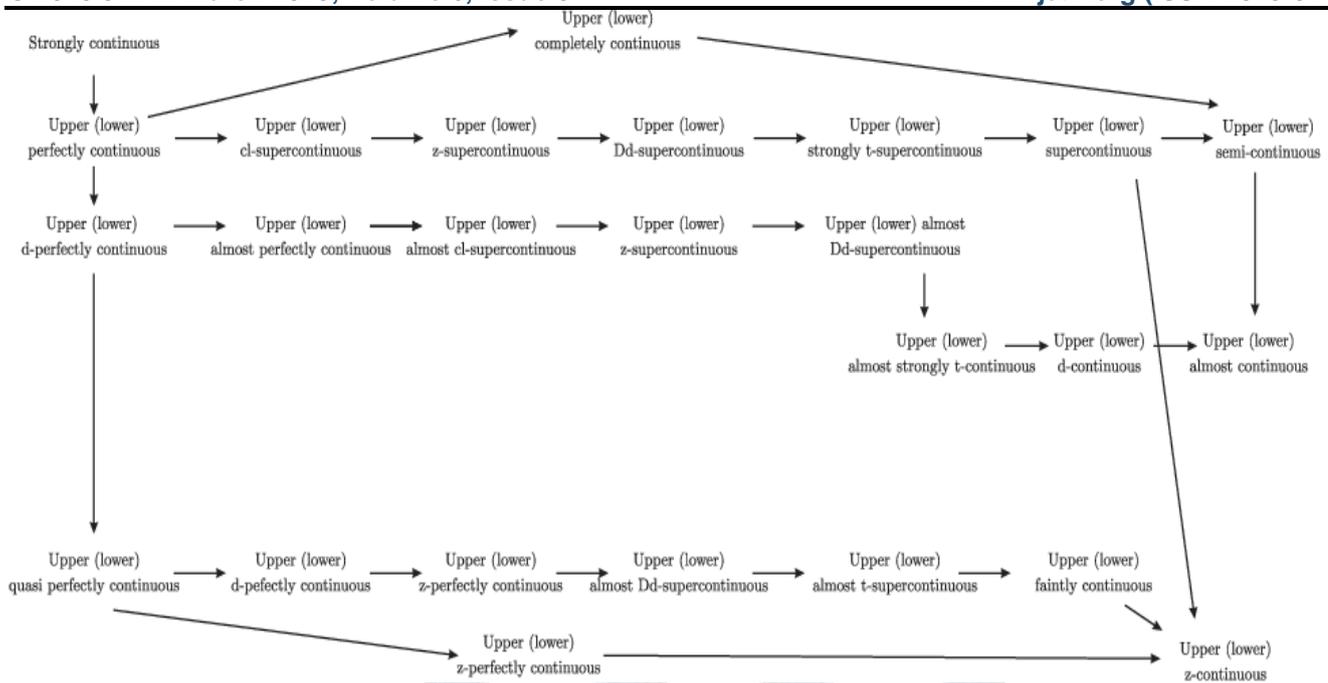


Figure 1

### Examples

**Example 3.1** Let  $X = \mathbb{R}$  with usual topology  $U$  and  $Y = \mathbb{R}$  with usual topology  $U$ . Let  $\varphi: (X, U) \rightarrow (Y, U)$  be the identity map  $\varphi(x) = x$  for all  $x \in X$ . Then  $\varphi$  is continuous as well as  $z$ -supercontinuous but not upper (lower)  $z$ -perfectly continuous.

### 4 Properties of upper $z$ -perfectly continuous multifunctions

**Proposition 4.1** If  $\varphi: X \rightarrow Y$  is upper  $z$ -perfectly continuous multifunction and  $\psi: Y \rightarrow Z$  is upper almost  $z$ -supercontinuous, then their composition  $\psi \circ \varphi$  is upper almost  $cl$ -supercontinuous.

*Proof.* Let  $V$  be a regular open set of  $Z$ . In view of upper almost  $z$ -super continuity of  $\psi$ ,  $\psi^{-1}(V)$  is cozero in  $Y$  and since  $\varphi$  is upper  $z$ -perfectly continuous, therefore  $\varphi^{-1}(\psi^{-1}(V))$  is clopen in  $X$ . Now  $(\psi \circ \varphi)^{-1}(V) = \varphi^{-1}(\psi^{-1}(V))$  is clopen in  $X$ . So  $\psi \circ \varphi$  is upper almost  $cl$ -supercontinuous.  $\square$

**Proposition 4.2** If  $\varphi: X \rightarrow Y$  be upper  $z$ -perfectly continuous multifunction and  $\psi: Y \rightarrow Z$  be upper quasi  $z$ -supercontinuous then their composition  $\psi \circ \varphi$  is upper quasi  $cl$ -supercontinuous.

*Proof.* Let  $V$  be a  $\theta$ -open set in  $Z$ . In view of upper quasi  $z$ -super continuity of  $\psi$ ,  $\psi^{-1}(V)$  is cozero in  $Y$  and since  $\varphi: X \rightarrow Y$  is upper  $z$ -perfectly continuous multifunction, therefore  $\varphi^{-1}(\psi^{-1}(V))$  is clopen in  $X$ .

Now  $(\psi \circ \phi)^{-1}(V) = \phi^{-1}(\psi^{-1}(V))$  is clopen in  $X$ .

Therefore  $\psi \circ \phi$  is upper quasi cl-supercontinuous.  $\square$

**Theorem 4.3** Let  $\varphi: X \rightarrow Y$  be multifunction. Then

- (a) If  $\varphi$  is upper  $z$ -perfectly continuous and  $A \subset X$  then the restriction  $\varphi|_A: A \rightarrow Y$  is upper  $z$ -perfectly continuous.
- (b) If  $Q = \{U_\alpha: \alpha \in \Lambda\}$  is a locally infinite clopen cover of  $X$  and if for each  $\alpha \in \Lambda$ , the restriction  $\varphi_\alpha = \varphi|_{U_\alpha}: U_\alpha \rightarrow Y$  is upper  $z$ -perfectly continuous then  $\varphi$  is upper  $z$ -perfectly continuous.

*Proof.* (a) Let  $W$  be a cozero set in  $Y$ . Since  $\varphi: X \rightarrow Y$  is upper  $z$ -perfectly continuous  $\varphi^{-1}(W)$  is clopen in  $X$ .

Now  $(\varphi|_A)^{-1}(W) = \{x \in A | \varphi(x) \subset W\} = \{x \in A | x \in \varphi^{-1}(W)\} = A \cap \varphi^{-1}(W)$  which is clopen in  $X$ .

So  $\varphi|_A$  is upper  $z$ -perfectly continuous.

(b) Let  $W$  be a cozero set in  $Y$ . Since  $\varphi_\alpha = \varphi|_{U_\alpha}: U_\alpha \rightarrow Y$  is upper  $z$ -perfectly continuous  $(\varphi_\alpha)^{-1}(W)$  is clopen set in  $U_\alpha$  and consequently clopen in  $X$ .

Since  $\varphi^{-1}(W) = \bigcup_{\alpha \in \Lambda} (\varphi_\alpha)^{-1}(W) = \bigcup_{\alpha \in \Lambda} (\varphi^{-1}(W) \cap U_\alpha)$ .

Now each  $\varphi^{-1}(W) \cap U_\alpha$  is clopen in  $U_\alpha$ , hence in  $X$ . Thus  $\varphi^{-1}(W)$  is open being the union of clopen sets. Moreover since collection  $Q$  is locally finite the collection  $\{\varphi^{-1}(W) \cap U_\alpha: \alpha \in \Lambda\}$  is locally finite collection of clopen sets. Since the union of locally finite collection of closed sets is closed,  $\varphi^{-1}(W)$  is also closed and hence clopen.

**Definition 4.4 ([32])** A topological space  $X$  is called an Alexandorff space if any intersection of open sets in  $X$  is open in  $X$ , equivalently any union of closed sets in  $X$  is closed in  $X$ .

Alexandorff spaces have been referred to as saturated spaces by Lorrain in [20].

**Theorem 4.5** For each  $\alpha \in \Lambda$ , let  $\varphi_\alpha: X \rightarrow X_\alpha$  be a multifunction and let  $\varphi: X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$  be defined by  $\varphi(x) = (\varphi_\alpha(x))$  for each  $x \in X$ . If  $\varphi$  is upper  $z$ -perfectly continuous, then each  $\varphi_\alpha$  is upper  $z$ -perfectly continuous.

Further, if  $X$  is an Alexandorff space and each  $\varphi_\alpha$  is upper  $z$ -perfectly continuous then  $\varphi$  is upper  $z$ -perfectly continuous.

*Proof.* Let  $\varphi$  be upper  $z$ -perfectly continuous. Now for each  $\alpha$ ,  $\varphi_\alpha = p_\alpha \circ \varphi$  where  $p_\alpha$  denotes the projection map since each projection map  $p_\alpha$  is continuous and hence upper  $z$ -continuous, in view of Theorem 4.3 it follows that each  $\varphi_\alpha$  is upper  $z$ -perfectly continuous.

Conversely, suppose that  $X$  is an Alexandorff space and each  $\varphi_\alpha$  is a upper  $z$ -perfectly continuous multifunction. Since  $X$  is Alexandorff, to show that the function  $\varphi$  is upper  $z$ -perfectly continuous, it is sufficient to show that  $\varphi^{-1}(S)$  is clopen for every subbasic cozero set  $S$  in the product space  $\prod_{\alpha \in \Lambda} X_\alpha$ . Let  $U_\beta \times \prod_{\alpha \in \Lambda} X_\alpha$  be a subbasic cozero set  $S$  in  $\prod_{\alpha \in \Lambda} X_\alpha$ , where  $U_\beta$  is cozero set in  $X_\beta$ . Then  $\varphi^{-1}(U_\beta \times \prod_{\alpha \neq \beta} X_\alpha) = \varphi^{-1}(p_\beta^{-1}(U_\beta)) = \varphi^{-1}(U_\beta)$  is clopen in  $X$  and so  $\varphi$  is upper  $z$ -perfectly continuous:  $\square$

## 5 Properties of lower $z$ -perfectly continuous multifunction

**Proposition 5.1** *If  $\varphi: X \rightarrow Y$  is lower  $z$ -perfectly continuous multifunction and  $\psi: Y \rightarrow Z$  is lower almost  $z$ -supercontinuous, then their composition  $\psi \circ \varphi$  is lower almost cl-supercontinuous.*

*Proof.* Let  $V$  be a regular open set of  $Z$ . In view of lower almost  $z$ -supercontinuity of  $\psi$ ,  $\psi_+^{-1}(V)$  is cozero in  $Y$  and since  $\varphi$  is lower  $z$ -perfectly continuous, therefore  $\varphi_+^{-1}(\psi_+^{-1}(V))$  is clopen in  $X$ . Now  $(\psi \circ \varphi)_+^{-1}(V) = \varphi_+^{-1}(\psi_+^{-1}(V))$  is clopen in  $X$ .

So  $\psi \circ \varphi$  is lower almost cl-supercontinuous.  $\square$

**Proposition 5.2** *If  $\varphi: X \rightarrow Y$  be lower  $z$ -perfectly continuous multifunction and  $\psi: Y \rightarrow Z$  be lower quasi  $z$ -supercontinuous then their composition  $\psi \circ \varphi$  is lower quasi cl-supercontinuous*

*Proof.* Let  $V$  be a  $\theta$ -open set in  $Z$ . In view of lower quasi  $z$ -supercontinuity of  $\psi$ ,  $\psi_+^{-1}(V)$  is cozero in  $Y$  and since  $\varphi: X \rightarrow Y$  is upper  $z$ -perfectly continuous multifunction, therefore  $\varphi_+^{-1}(\psi_+^{-1}(V))$  is clopen in  $X$ . Now  $(\psi \circ \varphi)_+^{-1}(V) = \varphi_+^{-1}(\psi_+^{-1}(V))$  is clopen in  $X$ .

Therefore  $\psi \circ \varphi$  is lower quasi cl-supercontinuous.  $\square$

**Theorem 5.3** *Let  $\varphi: X \rightarrow Y$  be a multifunction. Then*

- If  $\varphi$  is lower  $z$ -perfectly continuous and  $A \subset X$  then the restriction  $\varphi|_A: A \rightarrow Y$  is lower  $z$ -perfectly continuous:
- If  $Q = \{U_\alpha: \alpha \in \Lambda\}$  is a locally finite clopen cover of  $X$  and if for each  $\alpha \in \Lambda$ , the restriction  $\varphi_\alpha = \varphi|_{U_\alpha}: U_\alpha \rightarrow Y$  is lower  $z$ -perfectly continuous then  $\varphi$  is lower  $z$ -perfectly continuous.

*Proof.* (a) Let  $W$  be a cozero set in  $Y$ . Since  $\varphi: X \rightarrow Y$  is lower  $z$ -perfectly continuous,  $\varphi_+^{-1}(W)$  is clopen in  $X$ .

Now  $(\varphi|_A)_+^{-1}(W) = \{x \in A | \varphi(x) \cap W \neq \emptyset\} = \{x \in A | x \in \varphi_+^{-1}(W)\} = A \cap \varphi_+^{-1}(W)$ , which is clopen in  $X$  and so  $\varphi|_A$  is lower  $z$ -perfectly continuous.

(b) Let  $W$  be a cozero set in  $Y$ . Since  $\varphi_\alpha = \varphi|_{U_\alpha}: U_\alpha \rightarrow Y$  is lower  $z$ -perfectly continuous, so  $(\varphi_\alpha)_+^{-1}(W)$  is clopen set in  $U_\alpha$  and consequently clopen in  $X$ .

Since  $\varphi_+^{-1}(W) = \bigcup_{\alpha \in \Lambda} (\varphi_\alpha)_+^{-1}(W) = \bigcup_{\alpha \in \Lambda} (\varphi_+^{-1}(W) \cap U_\alpha)$ .

Now each  $\varphi_+^{-1}(W) \cap U_\alpha$  is clopen in  $U_\alpha$ , hence in  $X$ .

Thus  $\varphi_+^{-1}(W)$  is open being the union of clopen sets. Moreover since collection  $Q$  is locally finite, the collection  $\varphi_+^{-1}(W) \cap U_\alpha: \alpha \in \Lambda$  is locally finite collection of clopen set.

Since the union of locally finite collection of closed sets is closed,  $\varphi_+^{-1}(W)$  is also closed and hence clopen.  $\square$

**Theorem 5.4** For each  $\alpha \in \Lambda$ , let  $\varphi_\alpha: X \rightarrow X_\alpha$  be a multifunction and let  $\varphi: X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$  be defined by  $\varphi(x) = (\varphi_\alpha(x))$  for each  $x \in X$ . If  $\varphi$  is lower  $z$ -perfectly continuous, then each  $\varphi_\alpha$  is lower  $z$ -perfectly continuous. Further, if  $X$  is an Alexandorff space and each  $\varphi_\alpha$  is lower  $z$ -perfectly continuous then  $\varphi$  is lower  $z$ -perfectly continuous.

*Proof.* Let  $\varphi$  be lower  $z$ -perfectly continuous. Now for each  $\alpha$ ,  $\varphi_\alpha = p_\alpha \circ \varphi$  where  $p_\alpha$  denotes the projection map. Since each projection map  $p_\alpha$  is continuous and hence lower  $z$ -continuous, in view of Theorem 5.3 it follows that each  $\varphi_\alpha$  is lower  $z$ -perfectly continuous.

Conversely, suppose that  $X$  is an Alexandorff space and each  $\varphi_\alpha$  is a lower  $z$ -perfectly continuous multifunction. Since  $X$  is Alexandorff, to show that the function  $\varphi$  is lower  $z$ -perfectly continuous, it is sufficient to show that  $\varphi_+^{-1}(S)$  is clopen for every sub basic cozero set  $S$  in the product space  $\prod_{\alpha \in \Lambda} X_\alpha$ . Let  $U_\beta \times \prod_{\alpha \in \Lambda} X_\alpha$  be a subbasic cozero set in  $X_\beta$ . Then  $\varphi_+^{-1}(U_\beta \times \prod_{\alpha \neq \beta} X_\alpha) = \varphi_+^{-1}(p_\beta^{-1}(U_\beta)) = \varphi_+^{-1}(U_\beta)$  is clopen in  $X$  and so  $\varphi$  is lower  $z$ -perfectly continuous.  $\square$

## References

- [1] N. Ajmal and J.K. Kohli, Properties of hyperconnected spaces, their mappings into Hausdorff spaces and embeddings into hyperconnected spaces, Acta Math. Hungar. 60 (1-2) (1992), 41-49.

- [2] M.Akdağ, On the upper and lower super D-continuous multifunctions, Istanbul Univ. Sciences Faculty, Journal of Mathematics 60 (2001), 101-109.
- [3] M. Akdağ, On supercontinuous multifunctions, Acta Math. Hungar 99 (1-2) (2003), 143-153.
- [4] M. Akdağ on upper and lower  $z$ -supercontinuous multifunctions, Kyungpook Math. J. 45 (2005), 221-230.
- [5] M. Akdağ, On upper and lower  $D_\delta$ -supercontinuous multifunctions, Miskolc Mathematical Notes 7 (2006), 3-11.
- [6] P. Alexandorff, Discrete Roüme, Mat. Sb. 2 (1937), 501-518.
- [7] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Kluwer Academic Publishers, Dordrect, The Netherlands 1999.
- [8] N.C. Heldermaun, Developability and some new regularity axiom, Canad. J. Math. 33 (1981), 641-663.
- [9] L'. Holda V. Balaz and Neubrunn, Remarks on c-opeer semicontinuous multi-functions, Acta Mat. Univ. Comeniana 50/51 (1987), 159-165.
- [10] L'. Hola, Remarks on almost continuous multifunctions, Math. Slovaca 38 (1988), 325-331.
- [11] J.K. Kohli and C.P. Arya, Upper and lower cl-supercontinuous multifunctions, Applied General Topology (to appear).
- [12] J.K. Kohli and C.P. Arya, Strongly and perfectly continuous multifunctions, "Vasile Alescandri" University of Bacău, Faculty of Sciences, Scientific Studies and Research, Series Mathematics and Informatics, Vol. 20, No. 1 (2010), 103-118.
- [13] J.K. Kohli and C.P. Arya, Upper and lower almost cl-supercontinuous multi functions, Demonstratio Mathematica 44 (2) (2011), 407-421.
- [14] J.K. Kohli and C.P. Arya, Upper and lower (almost) completely continuous multifunctions, (preprint).
- [15] J.K. Kohli and C.P. Arya, Generalizations of  $z$ -supercontinuous and  $D_\delta$ -supercontinuous multifunctions, (preprint).
- [16] J.K. Kohli and R. Kumar,  $z$ -supercontinuous functions, Indian J. Pure Appl. Math. 33 (7) (2002), 1097-1108.
- [17] J.K. Kohli, C.P. Arya and D. Singh, Upper and lower Quasi-cl-supercontinuous, Multifunctions, Scientific studies and Research series Mathematics and Informatics 24 (2) (2010), 63-80.
- [18] J.K. Kohli and D. Singh, Almost cl-supercontinuous functions, Applied General Topology 10 (1) (2009), 1-12.

- [19] J.K. Kohli and D. Singh,  $\delta$ -perfectly continuous functions, *Demonstratio Mathematica* 42 (1) (2009), 221-231.
- [20] J.K. Kohli, D.Singh and C.P. Arya, Perfectly continuous functions, *Studii Si Cerctari Ser. Mat.* 18 (2008), 99-110.
- [21] J.K. Kohli D. Singh and R. Kumar, Quasi  $z$ -supercontinuous functions and pseudo  $z$ -supercontinuous functions, *Studii Si Cercetari Stintifice Seria Matematica Univ Bacau* 14 (2004), 43-56.
- [22] J.K. Kohli D.Singh and R. Kumar, Generalizations of  $z$ -supercontinuous functions and  $D_\delta$ -supercontinuous functions, *Appl. Gen. Top.* 9 (2) (2008), 239-251.
- [23] J.K. Kohli, D. Singh, J. Aggarwal and M.Rana, Pseudo perfectly continuous functions and closedness/compactness of their functions spaces, *Applied General Topology.* 14 (1) (2013), 115-134.
- [24] D. Singh and Manoj Kumar Rana,  $Z$ -perfectly continuous function and their function spaces, *Scientific Studies and Research series Mathematics and Informatics* 24 (2014), 135-152.
- [25] D. Singh, J.K. Kohli and B.K. Tyagi, Quasi perfectly continuous function and their function Spaces, *Scientific and Research Series Mathematics and Informatics* 21 (4) (2011), 23-46.
- [26] Y. Kucuk, On strongly  $\theta$ -continuous multifunctions, *Pure and Appl. Math. Sci.* 40 (194), 43-54.
- [27] N. Levine, Strong continuity in topological spaces, *Amer. Math. Monthly*, 67 (1960), 269.
- [28] F. Lorrain, Notes on topological spaces with minimum neighbourhood, *Amer. Math. Monthly* 76 (1967), 616-627.
- [29] J. Mack, Countable paracompactness and weak normality properties, *Trans Amer. Math. Soc.* 148 (1970), 265-272.
- [30] Manoj Kumar Rana, A new class of multifunction defined on cozero sets, *Journal of Emerging Technology and Innovative Research* Vol 5(8) (2018), 1164-1173.
- [31] Manoj Kumar Rana, New Properties of upper and lower  $z$ -perfectly continuous multifunctions, *Journal of Emerging Technology and Innovative Research* Vol 5(11) (2018), 195-203.
- [32] T. Noiri, On  $\delta$ -continuous functions, *J. Korean Math. Soc.* 16 (1980), 161-166.
- [33] T. Noiri, Supercontinuity and some strong forms of continuity, *Indian J. Pure Appl. Math.* 15 (3) (1984), 241-250.
- [34] W.J. Pervin and N. Levine, Connected mappings of Hausdorff spaces, *Proc Amer. Math. Soc.* 9 (3) (1958), 488-496.

- [35] I.L. Reilly and M.K. Vamanamurthy, On supercontinuous mappings, Indian J. Pure Appl. Math 14 (6) (1983), 767-772.
- [36] V. Popa and T. Noiri, On upper and Lower continuous multifunctions, Math Slovaca 43 (1993), 477-491.
- [37] N.V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl. 78 (2) (1986), 103-118.
- [38] R.E. Smitson, Multifunctions, Nieuw Archief Voor Wiskunde 3XX (1972), 31-53.

