ON SG-α-LOCALLY CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: The aim of this paper is to introduce and study the classes of sg-α-locally closed sets, sg-α-lc* sets and sg-α-lc** sets which are weaker forms of the class of locally closed sets. Furthermore the relations with other notions connected with the forms of locally closed sets are investigated.
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1. INTRODUCTION

The first step of locally closedness done by Bourbaki [1]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Stone [9] used the term FG for a locally closed set. Ganster and Reilly used locally closed sets in [2] to define LC-continuity and LC-irresoluteness. The aim of this paper is to introduce three forms of locally closed sets called sg-α-locally closed sets, sg-α-lc* sets and sg-α-lc** sets. Properties of these new concepts are also studied.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A, respectively. If $A \subseteq B \subseteq X$, then $cl_B(A)$ and $int_B(A)$ denote the closure of A relative to B and an interior of A relative to B.

We recall the following definitions, which are useful in the sequel.

Definition 2.1: A subset A of a space (X, τ) is called

(i). a semi-open set [3] if $A \subseteq cl(int(A))$ and

(ii). an α-open set [5] if $A \subseteq int(cl(int(A)))$.

The complement of α-closed set is called α-open. The α-closure [4] of a subset A of X, denoted by $\alpha cl_X(A)$ (briefly αcl(A)) is defined to be the intersection of all α-closed sets containing A.

Definition 2.2: A subset A of a topological space (X, τ) is called a semi-generalized α-closed (briefly sg-α-closed) set [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of sg-α-closed sets is called sg-α-open.

Definition 2.3: [7] Let (X, τ) be a topological space and $E \subseteq X$. We define the semi generalized-α-closure of E (briefly α^*-cl(E)) to be the intersection of all sg-α-closed sets containing E.

Definition 2.4: [2] A subset A of a space (X, τ) is called locally closed (briefly lc-set) set [14] if $A = C \cap D$, where C is open and D is closed in (X, τ).

Theorem 2.5: [7] An arbitrary intersection of sg-α-closed sets is sg-α-closed.

Lemma 2.6: [6] If $A \in SO(X_0)$, then $A = B \cap X_0$ for some $B \in SO(X)$, where X is a topological space and X_0 is a subspace of X.

3. SG-α-LOCALLY CLOSED SETS

We introduce the following definition

Definition 3.1: A subset A of (X, τ) is called sg-α-locally closed (briefly sg-α-lc) if $A = C \cap D$, where C is sg-α-open and D is sg-α-closed in (X, τ). The class of all sg-α-locally closed sets is denoted by SG-α-LC(X, τ).

Proposition 3.2: Every sg-α-closed (resp. sg-α-open) set is sg-α-locally closed.

Proof: The proof follows from the definitions.

Remark 3.3: The converse of this Proposition need not be true in general as seen from the following example.

Example 3.4: Let X = {a, b, c} and τ = {ϕ, {a}, X}. Then the set {a} is sg-α-lc but not sg-α-closed and the set {b, c} is sg-α-lc but not sg-α-open in (X, τ).

Proposition 3.5: Every locally closed set is sg-α-locally closed set.

Proof: The proof follows from the definitions.

Remark 3.6: The converse of the above Proposition 3.5 need not be true as seen from the following example.

Example 3.7: Let X = {a, b, c} and τ = {ϕ, {a, b}, X}. Then the set {a} is sg-α-lc but not lc set in (X, τ).

Definition 3.8: A subset A of a topological space (X, τ) is called

(1) sg-α-lc* if $A = U \cap F$, where U is sg-α-open in (X, τ) and F is closed in (X, τ),

(2) sg-α-lc** if $A = U \cap F$, where U is open in (X, τ) and F is sg-α-closed in (X, τ).

The class of all sg-α-lc*(resp. sg-α-lc**) sets in a topological space (X, τ) is denoted by SG-α-LC*(X, τ) (resp. SG-α-LC**(X, τ)).

Theorem 3.9: For a subset A of a topological space (X, τ), the following statements are equivalent.

(i). $A \in$ SG-α-LC(X, τ),

(ii). $A = U \cap \alpha^*$-cl(A) for some sg-α-open set U,

(iii). α^*-cl(A) \ A is $\tilde{g}$ -closed,

(iv). $A \cup (\alpha^*\text{-cl}(A))^c$ is sg-α-open,

(v). $A \subseteq \alpha^*\text{-int}(A \cup (\alpha^*\text{-cl}(A))^c)$.

Proof: (i) ⇒ (ii): Let $A \in$ SG-α-LC(X, τ). Then $A = U \cap F$ where U is sg-α-open and F is sg-α-closed in (X, τ). Since $A \subseteq F$, $\alpha^*\text{-cl}(A) \subseteq F$ and so $U \cap \alpha^*\text{-cl}(A) \subseteq A$. Also $A \subseteq U$ and $A \subseteq \alpha^*\text{-cl}(A)$ implies $A \subseteq U \cap \alpha^*\text{-cl}(A)$ and therefore $A = U \cap \alpha^*\text{-cl}(A)$.

(ii) ⇒ (iii): $A = U \cap \alpha^*\text{-cl}(A)$ implies $\alpha^*\text{-cl}(A) \setminus A = \alpha^*\text{-cl}(A) \cap U^c$ which is sg-α-closed since U^c is sg-α-closed.

(iii) ⇒ (iv): $A \cup (\alpha^*\text{-cl}(A))^c = (\alpha^*\text{-cl}(A) \setminus A)^c$ and by assumption, $(\alpha^*\text{-cl}(A) \setminus A)^c$ is sg-α-open and so is $A \cup (\alpha^*\text{-cl}(A))^c$.

(iv) ⇒ (v): By assumption, $A \cup (\alpha^*\text{-cl}(A))^c = \alpha^*\text{-int}(A \cup (\alpha^*\text{-cl}(A))^c)$ and hence $A \subseteq \alpha^*\text{-int}(A \cup (\alpha^*\text{-cl}(A))^c)$.

(v) ⇒ (i): By assumption and since $A \subseteq \alpha^*\text{-cl}(A)$, $A = \alpha^*\text{-int}(A \cup (\alpha^*\text{-cl}(A))^c) \cap \alpha^*\text{-cl}(A)$. Therefore, $A \in$ SG-α-LC(X, τ).

Theorem 3.10: For a subset A of (X, τ), the following are equivalent.

(i). $A \in$ SG-α-LC*(X, τ),

(ii). $A = U \cap \text{cl}(A)$ for some sg-α-open set U,

(iii). cl(A) \ A is sg-α-closed and

(iv). $A \cup (\text{cl}(A))^c$ is sg-α-open.

Proof: (i) ⇒ (ii): Let $A \in$ SG-α-LC*(X, τ). There exist a, sg-α-open set U and a closed set F such that $A = U \cap F$. Since $A \subseteq U$ and $A \subseteq \text{cl}(A)$, $A \subseteq U \cap \text{cl}(A)$. Also since $\text{cl}(A) \subseteq F$, $U \cap \text{cl}(A) \subseteq U \cap F = A$.

(ii) ⇒ (i): Since U is sg-α-open and cl(A) is a closed set, $A = U \cap \text{cl}(A) \in$ SG-α-LC*(X, τ).

(ii) ⇒ (iii): Since $\text{cl}(A) \setminus A = \text{cl}(A) \cap U^c$, cl(A) \ A is sg-α-closed by Corollary 3.22 [7].

(iii) ⇒ (ii): Let $U = (\text{cl}(A) \setminus A)^c$. Then by assumption U is sg-α-open in (X, τ) and $A = U \cap \text{cl}(A)$.

(iii) $\Rightarrow$ (iv): Let F = cl(A) \ A. Then $F^c = A\cup(cl(A))^c$ and $A\cup(cl(A))^c$ is sg-α-open.

(iv) $\Rightarrow$ (iii): Let $U = A\cup(cl(A))^c$. Then U^c is sg-α-closed and U^c = cl(A) \ A and so cl(A) \ A is sg-α-closed.

Theorem 3.11: Let A be a subset of (X, τ). Then A$\in$ SG-α-LC**(X, τ) if and only if A= U$\cap\alpha^*$-cl(A) for some open set U.

Proof: Necessity. Let A$\in$ SG-α-LC**(X, τ). Then A = U$\cap$F where U is open and F is sg-α-closed. Since A $\subseteq$ F, it follows α^*-cl(A) $\subseteq$ F. We obtain A = A$\cap\alpha^*$-cl(A) = U$\cap$F$\cap\alpha^*$-cl(A) = U$\cap\alpha^*$-cl(A).

Sufficiency: Obvious.

Corollary 3.12: Let A be a subset of (X, τ). If A$\in$ SG-α-LC**(X, τ), then α^*-cl(A) \ A is sg-α-closed and A$\cup(\alpha^*$-cl(A))^c is sg-α-open.

Proof: Let A$\in$ SG-α-LC**(X, τ). Then by Theorem 3.11, A = U$\cap\alpha^*$-cl(A) for some open set U and α^*-cl(A) \ A = α^*-cl(A)$\cap U^c$ is sg-α-closed in (X, τ). If F = α^*-cl(A) \ A , then F^c = A$\cup(\alpha^*$-cl(A))^c and F is sg-α-open and so is A$\cup(\alpha^*$-cl(A))^c.

Now we define sg-α-LC-continuous, sg-α-LC*-continuous and sg-α-LC**-continuous functions which are weaker than LC-continuous functions. We also define and study the respective irresolute functions.

Definition 3.13: A function f: (X, τ) $\to$ (Y, σ) is called sg-α-LC-continuous (resp. sg-α-LC*-continuous, sg-α-LC**-continuous) if f^{-1}(V)$\in$SG-α-LC(X, τ) (resp. f^{-1}(V)$\in$ SG-α-LC*(X, τ), f^{-1}(V)$\in$ SG-α-LC**(X, τ)) for every closed set V in (Y, σ).

Definition 3.14: A function f: (X, τ) $\to$ (Y, σ) is called sg-α-LC-irresolute (resp. sg-α-LC*-irresolute, sg-α-LC**-irresolute) if f^{-1}(V) $\in$ SG-α-LC(X, τ) (resp. f^{-1}(V)$\in$ SG-α-LC*(X, τ), f^{-1}(V)$\in$ SG-α-LC**(X, τ)) for every V $\in$ SG-α-LC(Y, σ) (resp. V$\in$ SG-α-LC*(Y, σ), V$\in$ SG-α-LC**(Y, σ)).

Proposition 3.15: Let f: (X, τ) $\to$ (Y, σ) be a function. Then we have the following:

(i). If f is sg-α-irresolute, then it is sg-α-LC-irresolute.

(ii). If f is LC-continuous, then it is sg-α-LC-continuous, sg-α-LC*-continuous and sg-α-LC**-continuous.

(iii). If f is sg-α-LC*-continuous or sg-α-LC**-continuous, then it is sg-α-LC-continuous.

Proof: (i). Let V$\in$ SG-α-LC(Y, σ). Then V = U$\cap$F for some sg-α-open set U and for some sg-α-closed set F in (Y, σ). We have $f^{-1}(V) = f^{-1}(U)\cap f^{-1}(F)\in$ SG-α-LC(X, τ), since f is sg-α-irresolute.

(ii). Follows from the fact that every locally closed set is sg-α-locally closed set, sg-α-lc* set and sg-α-lc** set.

(iii). Since every sg-α-lc* set is sg-α-locally closed and every sg-α-lc** set is sg-α-locally closed, the proof follows.

(iv). Follows from the fact that every open set is sg-α-locally closed set, sg-α-lc* set and sg-α-lc** set.

Remark 3.16: The converses of Proposition 3.15 need not be true in general as seen from the following examples.

Example 3.17: Let X = Y = {a, b, c}, τ = {∅, {a}, {b}, {a, b}, X} and σ = {∅, {a}, Y}. Define a map f: (X, τ) → (Y, σ) by f(a) = b, f(b) = c and f(c) = a. Then f is sg-α-LC-irresolute. However, f is not sg-α-irresolute, since for the sg-α-closed set U = {b, c} in (Y, σ), $f^{-1}(U)$ = {a, b} which is not sg-α-closed in (X, τ). Also the map f is sg-α-LC-continuous. However, f is not sg-α-irresolute, since for the sg-α-closed set U = {b, c} in (Y, σ), $f^{-1}(U)$ = {a, b} which is not sg-α-closed in (X, τ).

Example 3.18: Let X = Y = {a, b, c}, τ = {∅, {a, b}, X} and σ = {∅, {a, c}, Y}. Then the identity map f: (X, τ) → (Y, σ) is sg-α-LC-continuous. However, f is not sg-α-LC*-continuous, since for the open set U = {a, c} in (Y, σ), $f^{-1}(U)$ = {a, c}, which is not sg-α-lc*-set in (X, τ).

Proposition 3.19: Any map defined on a door space is sg-α-LC-continuous (resp. sg-α-LC-irresolute).

Proof: Let f: (X, τ) → (Y, σ) be a function, where (X, τ) be a door space and (Y, σ) be any space. Let A∈σ (resp. A∈ SG-α-LC(Y, σ)). Then by the assumption on (X, τ), $f^{-1}(A)$ is either open or closed. In both cases, $f^{-1}(A)$∈ SG-α-LC(X, τ) and therefore f is sg-α-LC-continuous (resp. sg-α-LC-irresolute).

Proposition 3.20: If f: (X, τ) → (Y, σ) be a sg-α-LC**-continuous (resp. sg-α-LC**-irresolute) and a subset B is sg-α-closed in (X, τ), then the restriction, f_B: (B, τ_B) → (Y, σ) is sg-α-LC**-continuous (resp. sg-α-LC**-irresolute).

Proof: Let G be an open (resp. sg-α-lc**) set of (Y, σ). Then $f^{-1}(G)$ = U∩F for some open set U and sg-α-closed set F of (X, τ). Then $f_B^{-1}(G)$ = (U∩B)∩(F∩B)∈ sg-α-LC**-(B, τ_B), by Theorem 2.5 and Lemma 2.6. Hence f_B is sg-α-LC**-continuous (resp. sg-α-LC**-irresolute).

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