Modified Thainwan's Iteration Procedure for Uniformly Continuous Total Asymptotically Nonexpansive Mappings in CAT(k) Spaces

Pinki, Research Scholar

Department of Mathematics, Maharshi Dayanand University,

Rohtak (Haryana) - 124001, India

Abstract

Our goal is to establish strong and Δ-convergence theorems of modified Thainwan's Iteration process for total asymptotically nonexpansive mapping in CAT(k) space with k > 0. Our results are the extensions and generalizations of several recent results from the current literature.

Keywords:

Fixed point, Total asymptotically nonexpansive mapping, CAT(k) space, Thainwan's Iteration, Δ-convergence, Strong convergence.

1. Introduction

Kirk([9], [10]) first studied the theory of fixed point in CAT(k) spaces. Later on, many authors generalized the notion of CAT(k) given in ([9], [10]), mainly focusing on CAT(0) spaces. The results of a CAT(0) space for every $k' \geq k$ (see in [3]). Although, CAT(k) spaces for k > 0, were studied by some authors (see [3], [7], [11]).

In 2008, Kirk and Panyanak [11] used the concept of Δ-convergence introduced by Lim[12] to prove the CAT(0) space analogs of some Banach space results which involve weak convergence.

Alber et al. [1] first introduced the total asymptotically nonexpansive mappings in Banach spaces. He generalizes the concept of asymptotically nonexpansive mappings. Recently, Panyanak [14] studied the existence theorem, demiclosed principle, Δ-convergence and strongly convergence theorems for uniformly continuous total asymptotically nonexpansive mapping in CAT(k) spaces. Moreover, there were many authors who have studied about this mapping (see [2], [5], [8]). This paper was organized as follows. In section 2 and 3, we present preliminaries and results of strong and Δ-convergence, respectively.

2. Preliminaries and Lemmas

In 2008, Thianwan defined the new two step iteration [17] as

$$x_{n+1} = \alpha_n T y_n \oplus (1-\alpha_n) y_n ,$$

$$y_n = \beta_n T x_n \oplus (1-\beta_n) x_n ,$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0, 1].

We now consider the modified Thainwan's iteration in the setting of CAT(k) spaces as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n) y_n , \\ y_n &= \beta_n T^n x_n \oplus (1-\beta_n) x_n , \end{aligned} \qquad (2.1)$$

Where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0, 1].

Let (X, ρ) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map γ from a closed interval [0, l] ⊂ R to X such that $\gamma(0)=x$, $\gamma(l)=y$, and $\rho(\gamma(t),\gamma(t'))=|t-t'|$ for all t, t' ∈ [0, l]. In particular, γ is an isometry and $\rho(x,y)=l$. The image $\gamma([0,l])$ of γ is called a geodesic segment joining x and y. When it is unique this geodesic segment is denoted [x, y]. This means that z ∈ [x, y] if and only if there exists $\alpha\in[0,1]$ such that

$$\rho(x,z)=(1-\alpha)\rho(x,y) \text{ and } \rho(y,z)=\alpha\rho(x,y).$$

In this case, we write $z=\alpha x\oplus(1-\alpha)y$. The space (X,ρ) is said to be a geodesic space (D-geodesic space) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (D-uniquely geodesic) if there exactly one geodesic joining x and y for each x, y ∈ X (for x, y ∈ X with $\rho(x,y)<D$). A subset K of X is said to be convex if K includes every geodesic segment joining any two of its points. The set K is said to be bounded if $diam(K)=\sup\{\rho(x,y):x,y\in K\}<\infty$.

Now we introduce the model spaces M_k^n, for more details on these spaces the reader is referred to [3].

Definition2.1. Given k ∈ R, we denote by M_k^n the following metric spaces:

1) If k = 0 then M_0^n is the Euclidean space E^n;
2) If k > 0 then M_k^n is obtained from the spherical space S^n by multiplying the distance function by the constant $1/\sqrt{k}$;
3) If k < 0 then M_k^n is obtained from the hyperbolic space H^n by multiplying the distance function by the constant $1/\sqrt{-k}$,

A geodesic triangle $\Delta(x,y,z)$ in a geodesic space (X,ρ) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x,y,z)$ in (X,ρ) is a triangle $\Delta(\bar{x},\bar{y},\bar{z})$ in M_k^2 such that

$$\rho(x,y)=d_{M_k^2}(\bar{x},\bar{y}),\ \rho(x,z)=d_{M_k^2}(\bar{x},\bar{z}) \text{ and } \rho(z,x)=d_{M_k^2}(\bar{z},\bar{x}).$$

If $k\le 0$ then such a comparison triangle always exists in M_k^2. If k > 0 then such a triangle exists whenever $\rho(x,y)+\rho(y,z)+\rho(z,x)<2D_k$, where $D_k=\pi/\sqrt{k}$. A point $\bar{p}\in[\bar{x},\bar{y}]$ is called a comparison point for $p\in[x,y]$ if $\rho(x,p)=d_{M_k^2}(\bar{x},\bar{p})$.

A geodesic triangle $\Delta(x,y,z)$ in X is said to satisfy the CAT(k) inequality if for any p, q∈ $\Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q}\in\Delta(\bar{x},\bar{y},\bar{z})$, one has

$$\rho(p,q)=d_{M_k^2}(\bar{p},\bar{q})$$

Definition 2.2. If $k\le 0$, then X is called a CAT(k) space if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality. If k > , then X is called a CAT(k) space if and only if X is D_k -geodesic and any geodesic triangle $\Delta(x,y,z)$ in X with $\rho(x,y)+\rho(y,z)+\rho(z,x)<2D_k$ satisfies the CAT(k) inequality.

Notice that in a CAT(0) space (X,ρ), if x, y, z ∈ X then the CAT(0) inequality implies

$$\rho^2\left(x,\frac{1}{2}y\oplus\frac{1}{2}z\right)\le\frac{1}{2}\rho^2(x,y)+\frac{1}{2}\rho^2(x,z)-\frac{1}{4}\rho^2(y,z). \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [4]. This inequality is extended by Dhompongsa and Panyanak [6] as

$$\rho^2(x,(1-\alpha)y\oplus\alpha z)\le(1-\alpha)\rho^2(x,y)+\alpha\rho^2(x,z)-\alpha(1-\alpha)\rho^2(y,z). \qquad \text{(CN}^*\text{)}$$

For all $\alpha\in[0,1]$ and x, y, z ∈ X.

Let $R\in(0,2]$. Recall that a geodesic space (X,ρ) is said to be R-convex for R(see [13]) if for any three points x, y, z ∈ X , we have

$$\rho^2(x,(1-\alpha)y\oplus\alpha z)\le(1-\alpha)\rho^2(x,y)+\alpha\rho^2(x,z)-\frac{R}{2}\alpha(1-\alpha)\rho^2(y,z). \quad (2.2)$$

The following lemma is a consequence of Proposition 3.1 in [13].

Lemma2.3. Let $k>0$ and (X,ρ) be a CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in(0,\pi/2)$. Then (X,ρ) is R-convex for $R=(\pi-2\varepsilon)\tan(\varepsilon)$.

The following lemma is also needed.

Lemma 2.4 ([3]). Let $k>0$ and (X,ρ) be a complete CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in(0,\pi/2)$. Then

$$\rho((1-\alpha)x\oplus\alpha y,z)\le(1-\alpha)\rho(x,z)+\alpha\rho(y,z), \text{ for all x, y, z} \in X \text{ and } \alpha\in[0,1]$$

We now collect some elementary facts about CAT(k) spaces. We state the results in CAT(k) with k > 0. Let $\{x_n\}$ be a bounded sequence in a CAT(k) space (X,ρ). For x ∈ X, we set

$$r(x,\{x_n\})=\limsup_{n\to\infty}\rho(x,\{x_n\}).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\})=\inf\{r(x,\{x_n\}):x\in X\}.$$

And the asymptotic centre $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\})=\{x\in X:r(x,\{x_n\})=r(\{x_n\})\}.$$

It is known that from [7] that in a CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of $\Delta-$convergence and collect some of its basic properties.

Definition2.5 ([11, 12]). A sequence $\{x_n\}$ in X is said to $\Delta-$converges to x ∈ X if x is the unique asymptotic centre of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta-\lim_n x_n=x$ and call x the $\Delta-$limit of $\{x_n\}$.

Lemma 2.6 ([15]). Let k > 0 and (X,ρ) be a complete CAT(k) Let $k>0$ and (X,ρ) be a complete CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in(0,\pi/2)$. Then the following statements hold:

i.every sequence in X has a $\Delta-$convergence subsequence;

ii.if $\{x_n\}\subseteq X$ and $\Delta-\lim_n x_n=x$, then $x\in\cap_{k=1}^{\infty}conv\{x_k,x_{k+1},....\}$,where

$$\mathrm{conv}(A)=\cap\{B:B\supseteq A \text{ and } B \text{ is closed and convex}\}.$$

By the uniqueness of asymptotic center, we can obtain the following lemma(see [6]).

Lemma 2.7. Let k > 0 and (X,ρ) be a complete CAT(k) Let $k>0$ and (X,ρ) be a complete CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in(0,\pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\})=\{x\}$ and let $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and the sequence $\{\rho(x_n,u)\}$ converges, then x = u.

Definition 2.8. Let K be a nonempty subset of a CAT(k) space (X,ρ). A mapping T: $K\to K$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n\to0,\mu_n\to0$ as

$n \to \infty$ and a strictly increasing continuous function $\psi : [0,1) \to [0,1)$ with $\psi(0) = 0$ such that

$\rho(T^n x, T^n y) \le \rho(x,y) + \nu_n \psi(\rho(x,y)) + \mu_n$ for all n ∈ N, x, y ∈ K.

A point x ∈ K is called a fixed point of T if x = T(x). We denote with F(T) the set of fixed points of T. A sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS in short) if

$\lim_{n\to\infty} \rho(x_n, T(x_n)) = 0$.

Lemma 2.9 ([16]). Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying $\{s_{n+1}\} \le \{s_n\} + \{t_n\}$ for all n ∈ N. If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n\to\infty} s_n$ exists.

Theorem 2.10 ([14]. Let k > 0 and (X, ρ) be a complete CAT(k) space with $diam(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T: K →K be a continuous total asymptotically nonexpansive mapping. Then T has a fixed point in K.

Theorem 2.11 ([14]). Let k > 0 and (X, ρ) be a complete CAT(k) space with $diam(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T : K →K be a uniformly continuous total asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta \to \lim_{n\to\infty} x_n = \omega$, then $\omega \in$ K and $\omega = T(\omega)$.

Definition 2.12. Let (X, ρ) be a metric space and K be its nonempty subset. Then T: K →K is said to be semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} \rho(x_n, T(x_n)) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in$ K.

The purpose of this paper was to prove strong and Δ-convergence of the modified Thainwan's two step iteration process for uniformly continuous total asymptotically nonexpansive mappings in CAT(k) spaces. Our results extend and improve the recent known results.

3. Main Results

Lemma 3.1 Let k > 0 and (X , ρ) be a complete CAT(k) space with diam(X) ≤ $\frac{\frac{\pi}{2} - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T: K → K be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1) where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0, 1] such that $\lim_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ $\lim_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then $\{x_n\}$ is an AFPS for T and $\lim_{n\to\infty} \rho(x_n, p)$ exists for all p ∈ F(T).

Proof. We divide the proof of this lemma into two steps.

Step 1 : We will prove that $\lim_{n\to\infty} \rho(x_n, p)$ exists.

It follows that therorem 2.10 that F(T) ≠ ϕ.Let p ∈ F(T) and M = diam(K). Since T is total asymptotically nonexpansive, by lemma 2.4 we have

$$\rho(y_n, p) = \rho((1-\beta_n)x_n \oplus \beta_n T^n x_n, p)$$
$$\le (1-\beta_n)\rho(x_n, p) + \beta_n \rho(T^n x_n, p)$$
$$= (1-\beta_n)\rho(x_n, p) + \beta_n \rho(T^n x_n, T^n p)$$
$$\le (1-\beta_n)\rho(x_n, p) + \beta_n \{\rho(x_n, p) + \nu_n \psi(M) + \mu_n\}$$

$$=\rho(x_n, p)+\nu_n\beta_n\psi(M)+\beta_n\mu_n$$

This implies that

$$\rho(x_{n+1}, p)=\rho((1-\alpha_n)y_n \oplus \alpha_n T^n y_n, p)$$
$$\leq (1-\alpha_n)\rho(y_n, p)+\alpha_n\rho(T^n y_n, T^n p)$$
$$\leq (1-\alpha_n)\rho(y_n, p)+\alpha_n\{\rho(y_n, p)+\nu_n\psi(M)+\mu_n\}=\rho(x_n, p)+\nu_n(\beta_n+\alpha_n)\psi(M)+\mu_n(\beta_n+\alpha_n)$$

Since $\sum_{n=1}^{\infty}\nu_n<1$ and $\sum_{n=1}^{\infty}\mu_n<1$, by lemma 2.9 $\lim_{n\to\infty}\rho(x_n, p)$ exists.

Step 2 : We will prove that $\lim_{n\to\infty}\rho(x_n, Tx_n)=0$ i.e. $\{x_n\}$ is an AFPS for T . Since $\{x_n\}$ is bounded, there exists R > 0 such that $\{x_n\},\{y_n\}\subset B_{R'}(p)$ for all $n\geq 1$ with $R'<D_k/2$. In view of (2.1), we have

$$\rho^2(y_n, p)=\rho^2((1-\beta_n)x_n \oplus \beta_n T^n x_n, p)$$
$$\leq (1-\beta_n)\rho^2(x_n, p)+\beta_n\rho^2(T^n x_n, T^n p)-\frac{R}{2}\beta_n(1-\beta_n)\rho^2(T^n x_n, x_n)$$
$$\leq (1-\beta_n)\rho(x_n, p)^2+\beta_n[\rho(x_n, p)+\nu_n\psi(M)+\mu_n]^2-\frac{R}{2}\beta_n(1-\beta_n)\rho^2(T^n x_n, x_n)$$
$$\leq \rho^2(x_n, p)+A\nu_n+B\mu_n-\frac{R}{2}\beta_n(1-\beta_n)\rho^2(T^n x_n, x_n) \qquad (3.1)$$

for some A, B > 0.

Finally, from (2.1) and using (3.1), we have

$$\rho^2(x_{n+1}, p)=\rho^2((1-\alpha_n)y_n \oplus \alpha_n T^n y_n, p)$$
$$\leq \alpha_n\rho^2(T^n y_n, T^n p)+(1-\alpha_n)\rho^2(y_n, p)-\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n y_n, y_n)$$
$$\leq \alpha_n[\rho(y_n, p)+\nu_n\psi(M)+\mu_n]^2+(1-\alpha_n)\rho^2(y_n, p)$$
$$\leq \alpha_n[\rho^2(y_n, p)+(\nu_n\psi(M)+\mu_n)^2+2\nu_n\psi(M)\mu_n\rho(y_n, p)]+(1-\alpha_n)\rho^2(y_n, p)$$
$$\leq \rho^2(y_n, p)+(\nu_n\psi(M)+\mu_n)^2+2\nu_n\psi(M)\mu_n\rho(y_n, p)$$

$$\rho^2(x_{n+1}, p)\leq \rho^2(x_n, p)+C\nu_n+D\mu_n-\frac{R}{2}\beta_n(1-\beta_n)\rho^2(T^n x_n, x_n) \qquad (3.2)$$

for some C, D > 0. This implies that

$$\frac{R}{2}\beta_n(1-\beta_n)\rho^2(T^n x_n, x_n)\leq \rho^2(x_n, p)-\rho^2(x_{n+1}, p)+C\nu_n+D\mu_n$$

Since $\sum_{n=1}^{\infty}\nu_n<\infty$ and $\sum_{n=1}^{\infty}\mu_n<\infty$, $\rho(x_n, p)<R'$, we have

$$\sum_{n=1}^{\infty}\frac{R}{2}\beta_n(1-\beta_n)\rho^2(T^n x_n, x_n)<\infty .$$

Hence by the fact $\lim_{n\to\infty}\beta_n(1-\beta_n)>0$, we have

$$\lim_{n\to\infty}\rho(T^n x_n, x_n)=0 . \qquad (3.3)$$

By the uniform continuity of *T*, we have

$$\lim_{n\to\infty}\rho(T^{n+1}x_n, Tx_n)=0 \qquad (3.4)$$

Again, note that by definitions of x_{n+1} and y_n , we have

$$\rho(x_n, x_{n+1})=\rho(x_n, \alpha_n T^n y_n \oplus (1-\alpha_n)y_n)$$
$$\leq (1-\alpha_n)\rho(x_n, y_n)+\alpha_n\rho(x_n, T^n y_n)$$
$$\leq \rho(x_n, y_n)+\rho(x_n, T^n x_n)+\rho(T^n x_n, T^n y_n)$$
$$\leq 2\rho(x_n, y_n)+\rho(x_n, T^n x_n)+\nu_n\psi(M)+\mu_n$$
$$=2\rho(x_n, \beta_n T^n x_n+(1-\beta_n)x_n)+\rho(x_n, T^n x_n)+\nu_n\psi(M)+\mu_n$$
$$\leq 2\beta_n\rho(x_n, T^n x_n)+0+\rho(x_n, T^n x_n)+\nu_n\psi(M)+\mu_n$$

$$\leq (2\beta_n+1)\rho(x_n,T^nx_n)+\nu_n\psi(M)+\mu_n \to 0 \text{ as } n\to\infty \quad (3.5)$$

By (3.3), (3.5) and the uniform continuity of T, we have

$$\rho(x_n,Tx_n)\leq \rho(x_n,x_{n+1})+\rho(x_{n+1},T^{n+1}x_{n+1})+\rho(T^{n+1}x_{n+1},T^{n+1}x_n)+\rho(T^{n+1}x_n,Tx_n)$$

$$\leq \rho(x_n,x_{n+1})+\rho(x_{n+1},T^{n+1}x_{n+1})+\rho(x_{n+1},x_n)+\nu_{n+1}\psi(M)+\mu_{n+1}+\rho(T^{n+1}x_n,Tx_n)\to 0 \text{ as } n\to\infty.$$

This completes the proof.

Now, we are in a position to prove the $\Delta-$convergence theorem.

Theorem 3.2. Let k > 0 and (X,ρ) be a complete CAT(k) space with $diam(X)\leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in(0,\pi/2)$. Let K be a nonempty closed convex subset of X, and T: $K\to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\nu_n<\infty$ and $\sum_{n=1}^{\infty}\mu_n<\infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] such that $\lim_{n\to\infty}\alpha_n(1-\alpha_n)>0$ and $\lim_{n\to\infty}\beta_n(1-\beta_n)>0$. Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

Proof. Let $w_w(\{x_n\}):=\bigcup A(\{u_n\})$ where union is taken for all sequences $\{u_n\}$ of $\{x_n\}$. We first show that $w_w(\{x_n\})\subseteq F(T)$. Let $u\in w_w(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By lemma 2.6, there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $\Delta-\lim_n v_n=v\in K$. By lemma 3.1 and theorem 2.11, we have v ϵ F(T). Since $\lim_n \rho(x_n,v)$ exists, so u=v by lemma 2.7. This shows that $w_w(\{x_n\})\subset F(T)$.

Next, we show that $\Delta-$converges to a point in F(T), it is sufficient to show that $w_w(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and let $A(\{x_n\})=\{x\}$. Since $u\in w_w(\{x_n\})\subseteq F(T)$, by lemma 3.1 $\lim_n \rho(x_n,u)$ exists. And by Lemma 2.7, we have x = u. This completes the proof.

Theorem 3.3. Let k > 0 and (X,ρ) be a complete CAT(k) space with $diam(X)\leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in(0,\pi/2)$. Let K be a nonempty closed convex subset of X, and T: $K\to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\nu_n<\infty$ and $\sum_{n=1}^{\infty}\mu_n<\infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] such that $\lim_{n\to\infty}\alpha_n(1-\alpha_n)>0$ and $\lim_{n\to\infty}\beta_n(1-\beta_n)>0$. Suppose that T^m is semi-compact for some m ϵ N. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.1, $\lim_n \rho(x_n,Tx_n)=0$. By definition 2.12, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p\in K$ such that $\lim_{j\to\infty}x_{n_j}=p$. Again, by the uniform continuity of T, we have

$$\rho(Tp,p)\leq\rho(Tp,Tx_{n_j})+\rho(Tx_{n_j},x_{n_j})+\rho(x_{n_j},p)\to 0 \text{ as } j\to\infty.$$

That is, $p\in F(T)$. By Lemma 3.1, $\lim_n \rho(x_n,u)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself.

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