Strong and Delta-Convergence Theorems for Modified M Iteration Process in CAT(k) Spaces

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Abstract

In this paper, we give strong and delta-convergence results of modified M iteration process in CAT(k) space with k > 0 by using the concept of delta-convergence introduced by Dhompongsa, Panyanak [On delta-convergence theorems in CAT(0) spaces, Comput. Math. Appl., 56(2008), 2572-2579.]. Our results extend and improve the corresponding recent results of Saipara et al. [16] (J. Nonlinear Sci. Appl., 2015, 965-975) and many others existing in the literature.

Keywords:

Fixed point, Total asymptotically nonexpansive mapping, CAT(k) space, M iteration, Delta-convergence, Strong convergence.

1. Introduction

Let (X,ρ) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map γ from a closed interval [0, l] ⊂ R to X such that $\gamma(0)=x$, $\gamma(l)=y$, and $\rho(\gamma(t),\gamma(t'))=|t-t'|$ for all t, t' ∈ [0, l]. In particular, γ is an isometry and $\rho(x,y)=l$. The image $\gamma([0,l])$ of γ is called a geodesic segment joining x and y. When it is unique this geodesic segment is denoted [x, y]. This means that z ∈ [x, y] if and only if there exists $\alpha \in [0,1]$ such that

$\rho(x,z)=(1-\alpha)\rho(x,y)$ and $\rho(y,z)=\alpha\rho(x,y)$.

In this case, we write $z=\alpha x \oplus (1-\alpha) y$. The space (X,ρ) is said to be a geodesic space (D-geodesic space) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (D-uniquely geodesic) if there exactly one geodesic joining x and y for each x, y ∈ X (for x, y ∈ X with $\rho(x,y)<D$). A subset K of X is said to be convex if K includes every geodesic segment joining any two of its points. The set K is said to be bounded if $diam(K)=\sup\{\rho(x,y): x,y\in K\}<\infty$.

Kirk([9], [10]) first studied the theory of fixed point in CAT(k) spaces. Later on, many authors generalized the notion of CAT(k) given in ([9], [10]), mainly focusing on CAT(0) spaces. The results of a CAT(0) space for every $k' \geq k$ (see in [3]). Although, CAT(k) spaces for k > 0, were studied by some authors (see [3], [7], [11]).

Alber et al. [1] first introduced the total asymptotically nonexpansive mappings in Banach spaces. He generalizes the concept of asymptotically nonexpansive mappings. Recently, Panyanak [14] studied the existence theorem, demiclosed principle, Δ-convergence and strongly convergence theorems for uniformly continuous total asymptotically nonexpansive mapping in CAT(k) spaces. Moreover, there were many authors who have studied about this mapping (see [2], [5], [8]).

Ullah and Arshad [18] introduced a new three-step iteration process known as "M Iteration Process" defined as:

$$z_n=(1-\alpha_n)x_n+\alpha_n T x_n$$

$$y_n=Tz_n$$

$x_{n+1} = Ty_n$, $n \geq 0$ where $\{\alpha_n\}$ is a real sequence in [0, 1].

We now consider the modified M iteration process in the setting of CAT(k) spaces defined as:

$$z_n = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n$$

$$y_n = T^n z_n \tag{1.1}$$

$x_{n+1} = T^n y_n$, $n \geq 0$ where $\{\alpha_n\}$ is a real sequence in [0, 1].

Now we introduce the model spaces M_k^n , for more details on these spaces the reader is referred to [3].

Definition1.1. Given k ∈ R, we denote by M_k^n the following metric spaces:

1) If k = 0 then M_0^n is the Euclidean space E^n ;
2) If k > 0 then M_k^n is obtained from the spherical space S^n by multiplying the distance function by the constant $1/\sqrt{k}$;
3) If k < 0 then M_k^n is obtained from the hyperbolic space H^n by multiplying the distance function by the constant $1/\sqrt{-k}$,

A geodesic triangle $\Delta(x, y, z)$ in a geodesic space (X, ρ) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x, y, z)$ in (X, ρ) is a triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ in M_k^2 such that

$$\rho(x, y) = d_{M_k^2}(\bar{x}, \bar{y}),\ \rho(x, z) = d_{M_k^2}(\bar{x}, \bar{z}) \text{ and } \rho(z, x) = d_{M_k^2}(\bar{z}, \bar{x}).$$

If $k \leq 0$ then such a comparison triangle always exists in M_k^2. If k > 0 then such a triangle exists whenever $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_k$, where $D_k = \pi/\sqrt{k}$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $\rho(x, p) = d_{M_k^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\Delta(x, y, z)$ in X is said to satisfy the CAT(k) inequality if for any p, q ∈ $\Delta(x, y, z)$ and for their comparison points $\bar{p}, \bar{q} \in \Delta(\bar{x}, \bar{y}, \bar{z})$, one has

$$\rho(p, q) = d_{M_k^2}(\bar{p}, \bar{q})$$

Definition 1.2. If $k \leq 0$, then X is called a CAT(k) space if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality. If k > , then X is called a CAT(k) space if and only if X is D_k -geodesic and any geodesic triangle $\Delta(x, y, z)$ in X with $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_k$ satisfies the CAT(k) inequality.

Notice that in a CAT(0) space (X, ρ), if x, y, z ∈ X then the CAT(0) inequality implies

$$\rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho^2(x, y) + \frac{1}{2}\rho^2(x, z) - \frac{1}{4}\rho^2(y, z). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [4]. This inequality is extended by Dhompongsa and Panyanak [6] as

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \alpha(1-\alpha)\rho^2(y, z). \tag{CN*}$$

For all $\alpha \in [0,1]$ and x, y, z ∈ X.

Let $R \in (0, 2]$. Recall that a geodesic space (X, ρ) is said to be R-convex for R(see [13]) if for any three points x, y, z ∈ X , we have

$$\rho^2\left(x,(1-\alpha)y\oplus\alpha z\right)\le(1-\alpha)\rho^2\left(x,y\right)+\alpha\rho^2\left(x,z\right)-\frac{R}{2}\alpha(1-\alpha)\rho^2\left(y,z\right). \quad (1.2)$$

The following lemma is a consequence of Proposition 3.1 in [13].

Lemma 1.3. Let $k>0$ and $\left(X,\rho\right)$ be a CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in\left(0,\pi/2\right)$. Then $\left(X,\rho\right)$ is R-convex for $R=\left(\pi-2\varepsilon\right)\tan\left(\varepsilon\right)$.

The following lemma is also needed.

Lemma 1.4 ([3]). Let $k>0$ and $\left(X,\rho\right)$ be a complete CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in\left(0,\pi/2\right)$. Then

$$\rho\left((1-\alpha)x\oplus\alpha y,z\right)\le(1-\alpha)\rho\left(x,z\right)+\alpha\rho\left(y,z\right),$$ for all x, y, z $\in$ X and $\alpha\in[0,1]$

We now collect some elementary facts about CAT(k) spaces. We state the results in CAT(k) with k > 0. Let $\left\{x_n\right\}$ be a bounded sequence in a CAT(k) space $\left(X,\rho\right)$. For x $\in$ X, we set

$$r\left(x,\left\{x_n\right\}\right)=\limsup_{n\to\infty}\rho\left(x,\left\{x_n\right\}\right).$$

The asymptotic radius $r\left(\left\{x_n\right\}\right)$ of $\left\{x_n\right\}$ is given by

$$r\left(\left\{x_n\right\}\right)=\inf\left\{r\left(x,\left\{x_n\right\}\right):x\in X\right\}.$$

And the asymptotic centre $A\left(\left\{x_n\right\}\right)$ of $\left\{x_n\right\}$ is the set

$$A\left(\left\{x_n\right\}\right)=\left\{x\in X:r\left(x,\left\{x_n\right\}\right)=r\left(\left\{x_n\right\}\right)\right\}.$$

It is known that from [7] that in a CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$, $A\left(\left\{x_n\right\}\right)$ consists of exactly one point. We now give the concept of $\Delta-$convergence and collect some of its basic properties.

Definition1.5 ([11, 12]). A sequence $\left\{x_n\right\}$ in X is said to $\Delta-$converges to x $\in$ X if x is the unique asymptotic centre of $\left\{u_n\right\}$ for every subsequence $\left\{u_n\right\}$ of $\left\{x_n\right\}$. In this case we write $\Delta-\lim_n x_n=x$ and call x the $\Delta-$limit of $\left\{x_n\right\}$.

Lemma 1.6 ([15]). Let k > 0 and $\left(X,\rho\right)$ be a complete CAT(k) Let $k>0$ and $\left(X,\rho\right)$ be a complete CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in\left(0,\pi/2\right)$. Then the following statements hold:

i.every sequence in X has a $\Delta-$convergence subsequence;

ii.if $\left\{x_n\right\}\subseteq X$ and $\Delta-\lim_n x_n=x$, then $x\in\cap_{k=1}^{\infty}conv\{x_k,x_{k+1},....\}$,where

$$\mathrm{conv}(A)=\cap\{B:B\supseteq A \text{ and } B \text{ is closed and convex}\}.$$

By the uniqueness of asymptotic center, we can obtain the following lemma(see [6]).

Lemma 1.7. Let k > 0 and $\left(X,\rho\right)$ be a complete CAT(k) Let $k>0$ and $\left(X,\rho\right)$ be a complete CAT(k) space with $diam(X)\le\frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon\in\left(0,\pi/2\right)$. If $\left\{x_n\right\}$ is a sequence in X with $A\left(\left\{x_n\right\}\right)=\left\{x\right\}$ and let $\left\{u_n\right\}$ is a subsequence of $\left\{x_n\right\}$ with $A\left(\left\{u_n\right\}\right)=\left\{u\right\}$ and the sequence $\left\{\rho\left(x_n,u\right)\right\}$ converges, then x = u.

Definition 1.8. Let K be a nonempty subset of a CAT(k) space $\left(X,\rho\right)$. A mapping T: $K\to K$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\left\{\nu_n\right\}$, $\left\{\mu_n\right\}$ with $\nu_n\to0,\mu_n\to0$ as $n\to\infty$ and a strictly increasing continuous function $\psi:[0,1)\to[0,1)$ with $\psi\left(0\right)=0$ such that

$\rho\left(T^n x, T^n y\right) \le \rho\left(x, y\right) + \nu_n \psi\left(\rho\left(x, y\right)\right) + \mu_n$ for all n $\in$ N, x, y $\in$ K.

A point x $\in$ K is called a fixed point of T if x = T(x). We denote with F(T) the set of fixed points of T. A sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS in short) if

$\lim_{n\to\infty} \rho(x_n, T(x_n)) = 0$.

Lemma 1.9 [17]. Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying $\{s_{n+1}\} \le \{s_n\} + \{t_n\}$ for all n $\in$ N. If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n\to\infty} s_n$ exists.

Theorem 1.10 [14]. Let k > 0 and (X, ρ) be a complete CAT(k) space with $diam(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T: K $\to$K be a continuous total asymptotically nonexpansive mapping. Then T has a fixed point in K.

Theorem 1.11 [14]. Let k > 0 and (X, ρ) be a complete CAT(k) space with $diam(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$.Let K be a nonempty closed convex subset of X, and T : K $\to$K be a uniformly continuous total asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta \to \lim_{n\to\infty} x_n = \omega$, then $\omega \in$ K and $\omega = T(\omega)$.

Definition 1.12. Let (X, ρ) be a metric space and K be its nonempty subset. Then T: K $\to$K is said to be semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} \rho(x_n, T(x_n)) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in$K.

We now establish the following results:

2. Main Results

Lemma 2.1 Let k > 0 and (X , ρ) be a complete CAT(k) space with diam(X) ≤ $\frac{\frac{\pi}{2} - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T: K → K be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} v_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by $x_1 \in K$ and

$$z_n = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n$$

$$y_n = T^n z_n$$

$$x_{n+1} = T^n y_n, \ n \ge 0 \text{ where } \{\alpha_n\} \text{ is a real sequence in [0, 1].}$$

where $\{\alpha_n\}$ is sequence in [0, 1] such that $\lim_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then $\{x_n\}$ is an AFPS for T and $\lim_{n\to\infty} \rho(x_n, p)$ exists for all p ∈ F(T).

Proof. We divide the proof of this lemma into two steps.

Step 1 : We will prove that $\lim_{n\to\infty} \rho(x_n, p)$ exists.

It follows that therorem 2.10 that F(T) ≠ ϕ.Let p ∈ F(T) and M = diam(K). Since T is total asymptotically nonexpansive, by lemma 2.4 we have

$$\rho(z_n, p) = \rho\left((1 - \alpha_n)x_n \oplus \alpha_n T^n x_n, p\right)$$

$$\le (1 - \alpha_n)\rho(x_n, p) + \alpha_n \rho\left(T^n x_n, p\right)$$

$$=\left(1-\alpha_n\right)\rho(x_n,\,p)+\alpha_n\rho\left(T^n x_n,\,T^n p\right)$$

$$\le\left(1-\alpha_n\right)\rho(x_n,\,p)+\alpha_n\left\{\rho(x_n,\,p)+\nu_n\psi(M)+\mu_n\right\}$$

$$=\rho(x_n,\,p)+\alpha_n\nu_n\psi(M)+\alpha_n\mu_n.$$

This implies that

$$\rho(y_n,\,p)=\rho\left(T^n z_n,\,p\right)$$

$$=\rho\left(T^n z_n,\,T^n p\right)$$

$$\le\rho\left(z_n,\,p\right)+\nu_n\psi\left(M\right)+\mu_n$$

$$\le\rho\left(x_n,\,p\right)+(\alpha_n+1)\nu_n\psi\left(M\right)+(\alpha_n+1)\mu_n$$

$$\rho(x_{n+1},p)=\rho(T^n y_n,p)$$

$$=\rho(T^n y_{n,}T^n p)$$

$$\le\rho(y_{n,}p)+\nu_n\psi(M)+\mu_n$$

$$=\rho(x_n,p)+\nu_n(2+\alpha_n)\psi(M)+\mu_n(2+\alpha_n)$$

Since $\sum_{n=1}^{\infty}\nu_n<1$ and $\sum_{n=1}^{\infty}\mu_n<1$, by lemma 2.9 $\lim_{n\to\infty}\rho(x_n,p)$ exists.

Step 2 : We will prove that $\lim_{n\to\infty}\rho\left(x_n,Tx_n\right)=0$.

Next, we show that $\left\{x_n\right\}$ is an AFPS for T . Since $\left\{x_n\right\}$ is bounded, there exists R > 0 such that $\left\{x_n\right\}$, $\left\{y_n\right\}$, $\left\{z_n\right\}$ $\subset B_R^{'}(p)$ for all $n\ge1$ with $R'<D_k/2$. In view of (1.2), we have

$$\rho^2(z_n,p)=\rho^2((1-\alpha_n)x_n\oplus\alpha_n T^n x_n,p)$$

$$\le\alpha_n\rho^2(T^n x_n,p)+(1-\alpha_n)\rho^2(x_n,p)-\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n,x_n)$$

$$=\alpha_n\rho^2(T^n x_n,T^n p)+(1-\alpha_n)\rho^2(x_n,p)-\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n,x_n)$$

$$\le\alpha_n[\rho(x_n,p)+\nu_n\psi(M)+\mu_n]^2+(1-\alpha_n)\rho^2(x_n,p)-\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n,x_n)$$

$$=\alpha_n\rho^2(x_n,p)+(1-\alpha_n)\rho^2(x_n,p)+A\nu_n+B\mu_n-\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n,x_n)$$

$$=\rho^2(x_n,p)+A\nu_n+B\mu_n-\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n,x_n)\qquad(3.1)$$

For some A, B > 0. This implies that

$$\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n, x_n) \le \rho^2(x_n, p) - \rho^2(x_{n+1}, p) + A\nu_n + B\mu_n$$

Since $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$, $\rho(x_n, p) < R'$, we have

$$\sum_{n=1}^{\infty}\frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n x_n, x_n) < \infty .$$

Hence by the fact $\lim_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty}\rho(T^n x_n, x_n) = 0 . \qquad (3.2)$$

By the uniform continuity of T, we have

$$\lim_{n\to\infty}\rho(T^{n+1}x_n, Tx_n) = 0 \qquad (3.3)$$

By definitions of x_{n+1} and y_n , we have

$$\begin{aligned}
\rho(x_n, x_{n+1}) &= \rho(x_n, T^n y_n) \\
&\le \rho(x_n, T^n x_n) + \rho(T^n x_n, T^n y_n) \\
&\le \rho(x_n, T^n x_n) + \rho(x_n, y_n) + \nu_n\psi(M) + \mu_n \\
&= \rho(x_n, T^n x_n) + \rho(x_n, T^n z_n) + \nu_n\psi(M) + \mu_n \\
&\le \rho(x_n, T^n x_n) + \rho(x_n, T^n x_n) + \rho(T^n x_n, T^n z_n) + \nu_n\psi(M) + \mu_n \\
&= 2\rho(x_n, T^n x_n) + \rho(x_n, z_n) + 2\nu_n\psi(M) + 2\mu_n \\
&= 2\rho(x_n, T^n x_n) + \rho(x_n, (1-\alpha_n)x_n \oplus \alpha_n T^n x_n) + 2\nu_n\psi(M) + 2\mu_n \\
&\le 2\rho(x_n, T^n x_n) + \alpha_n\rho(x_n, T^n x_n) + (1-\alpha_n)\rho(x_n, x_n) + 2\nu_n\psi(M) + 2\mu_n \\
&\le (2+\alpha_n)\rho(x_n, T^n x_n) + 2\nu_n\psi(M) + 2\mu_n \to 0 \text{ as } n\to\infty
\end{aligned} \qquad (3.4)$$

By (3.2), (3.4) and the uniform continuity of T, we have

$$\begin{aligned}
\rho(x_n, Tx_n) &\le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, T^{n+1}x_{n+1}) + \rho(T^{n+1}x_{n+1}, T^{n+1}x_n) + \rho(T^{n+1}x_n, Tx_n) \\
&\le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, T^{n+1}x_{n+1}) + \rho(x_{n+1}, x_n) + \nu_{n+1}\psi(M) + \mu_{n+1} + \rho(T^{n+1}x_n, Tx_n) \to 0 \text{ as}
\end{aligned}$$

$n\to\infty$.

This completes the proof.

Now, we are in a position to prove the $\Delta-$convergence theorem.

Theorem 2.2. Let k > 0 and (X,ρ) be a complete CAT(k) space with $diam(X) \le \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T: $K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (1.1) where $\{\alpha_n\}$ is sequence in [0, 1] such that $\lim_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Then $\{x_n\}$ $\Delta-$converges to a fixed point of T.

Proof. Let $w_w(\{x_n\}) := \bigcup A(\{u_n\})$ where union is taken for all sequences $\{u_n\}$ of $\{x_n\}$. We first show that $w_w(\{x_n\}) \subseteq F(T)$. Let $u \in w_w(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By lemma 2.6, there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $\Delta - \lim_n v_n = v \in K$. By lemma 3.1 and theorem 2.11, we have v ϵ F(T). Since $\lim_n \rho(x_n, v)$ exists, so u=v by lemma 2.7. This shows that $w_w(\{x_n\}) \subset F(T)$.

Next, we show that $\Delta -$ converges to a point in F(T), it is sufficient to show that $w_w(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_w(\{x_n\}) \subseteq F(T)$, by lemma 3.1 $\lim_n \rho(x_n, u)$ exists. And by Lemma 2.7, we have x = u. This completes the proof.

Now we prove strong convergence theorem.

Theorem 3.3. Let k > 0 and (X, ρ) be a complete CAT(k) space with $diam(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and T: $K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} v_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (1.1) where $\{\alpha_n\}$ is sequence in [0, 1] such that $\lim_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose that T^m is semi-compact for some m ϵ N. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.1, $\lim_n \rho(x_n, Tx_n) = 0$. By definition 2.12, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of T, we have

$$\rho(Tp, p) \le \rho(Tp, Tx_{n_j}) + \rho(Tx_{n_j}, x_{n_j}) + \rho(x_{n_j}, p) \to 0 \text{ as } j \to \infty.$$

That is, $p \in F(T)$. By Lemma 3.1, $\lim_n \rho(x_n, u)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself.

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