πgb-NORMAL SPACES IN TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to introduce a new class of normal spaces, called πgb-normal spaces by using πgb-closed and b-open sets. The relationships among πgp-normal, πgβ-normal, πgb-normal, p-normal, β-normal, γ-normal, π-normal, πp-normal, πγ-normal, πβ-normal, almost normal, almost p-normal, almost γ-normal, almost β-normal, quasi normal, quasi p-normal, quasi γ-normal, quasi β-normal, mildly normal, mildly p-normal, mildly γ-normal and mildly β-normal spaces are investigated. We also prove that πgb-normality is a topological property and it is a hereditary property with respect to π-open, πgb-closed subspaces. Further we obtain a characterization and some preservation theorems for πgb-normal spaces.

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1. Introduction

In 1958, Kuratowski [**14**] introduced the concepts of regular open and regular closed sets in topological spaces. In 1968, Zaitsev [**39**] introduced the notion of quasi-normal spaces and obtained some characterizations and preservation theorems for quasi-normal spaces. In 1970, Levine [**16**] defined generalized closed sets in topological spaces. In 1970, Singal and Arya [**32**] introduced the notion of almost normal spaces and obtained their characterizations. In 1973, Singal and Singal [**33**] introduced the concept of mildly normal spaces and obtained their properties. In 1989, Nour [**24**] introduced the notion of p-normal spaces and obtained their characterizations and preservation theorems for p-normal spaces. In 1990, Mahmoud and Monsef [**17**] introduced the concept of β-normal spaces. In 2007, Ekici [**8**] introduced the concept of γ-normal spaces and obtained their characterizations and preservation theorems for γ-normal spaces. In 2008, Kalantan [**9**] introduced the notion of π-normal spaces and obtained some characterizations. In 2010, Tahiliani [**35**] introduced the notion of πgβ-closed sets and their properties are studied. In 2010, M. C. Sharma and Hamant Kumar [**29**] introduced the notion of πβ-normal spaces and obtained their characterizations. In 2012, Thabit and Kamaruihaili [**36**] introduced the notion of πp-normal spaces and obtained their characterizations. In 2012, Thabit and Kamaruihaili [**37**] introduced a weaker form of p-normality called quasi p-normality, which lies between πp-normality and mild p-normality. In 2013, Thanh and Thinh [**38**] introduced the notion of πgp-normal spaces and prove that πgp-normality is a topological property and it is a hereditary property with respect to π-open, πgp-closed subspaces. In 2015, M. C. Sharma and Hamant Kumar [**31**] introduced the concept of softly normal spaces and obtained their characterizations. In 2016, Hamant Kumar and M. C. Sharma [**12**] introduced the notions of almost γ-normal and πγ-normal spaces and obtained their characterizations. In 2016, Hamant Kumar and M. C.

Sharma [**11**] introduced the concepts of quasi γ-normal and mildly γ-normal spaces and obtained their properties.

2. Preliminaries

Throughout in this paper, the spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively.

2.1 Definition. A subset A of a space X is said to be
(1) **regular open** [**14**] if A = int(cl(A)).
(2) The finite union of regular open sets is said to be **π-open [39]**.
(3) **b-open [4]** if A ⊂ cl(int(A)) ∪ int(cl(A)).
(4) **p-open [20] if** A ⊂ int(cl(A)).
(5) **s-open [15] if** A ⊂ cl(int(A)).
(6) **α-open [22] if** A ⊂ int(cl(int(A))).
(7) **β-open [1] if** A ⊂ cl(int(cl(A))).
The complement of a regular open (resp. π-open, b–open, p–open, s–open, α–open β–open) set is said to be **regular closed** (resp. **π-closed, b–closed, p–closed, s–closed, α–closed, β-closed**).
The intersection of all b–closed (resp. p–closed, s–closed, α–closed, β-closed) sets containing A is called **b–closure** (resp. **p–closure, s–closure, α–closure, β-closure**) of A, and is denoted by **bcl(A)** (resp. **pcl(A), scl(A), αcl(A), βcl(A)**). The **b-Interior** of A, denoted by bint(A), is defined as union of all b-open sets contained in A.

2.2 Definition. A subset A of a space X is said to be
(1) generalized closed (briefly **g-closed**) [**16]** if cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(2) generalized pre-closed (briefly **gp-closed**) [**23])** if pcl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(3) generalized semi-closed (briefly **gs-closed**) [**3])** if scl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(4) α-generalized closed (briefly **αg-closed**) [**18])** if αcl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(5) **generalized β- closed** (briefly **gβ-closed**) [**6**]) if βcl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(6) **generalized b-closed** (briefly **gb-closed**) [**2**]) if bcl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(7) **πg-closed** [**7**] if cl(A) ⊂ U whenever A ⊂ U and U is π–open in X.
(8) **πgp-closed** [**25**] if pcl(A) ⊂ U whenever A ⊂ U and U is π–open in X.
(9) **πgs-closed** [**5**] if scl(A) ⊂ U whenever A ⊂ U and U is π–open in X.
(10) **πgα-closed** [**27**] if αcl(A) ⊂ U whenever A ⊂ U and U is π–open in X.
(11) **πgβ-closed** [**35**] if βcl(A) ⊂ U whenever A ⊂ U and U is π–open in X.
(12) **πgb-closed** [**34**] if bcl(A) ⊂ U whenever A ⊂ U and U is π–open in X.
(13) **g-open** (resp**. gp-open, gs-open, αg-open, gβ-open**, **gb-open**, **πg-open, πgp-open, πgs-open, πgα-open, πgβ-open, πgb-open**) if the complement of A is g–closed (resp. gp–closed, gs–closed, αg–closed gβ-closed, gb-closed, πg–closed, πgp–closed, πgs–closed, πgα–closed, πgβ–closed, πgb–closed).

Clearly, from above definitions, we have the following diagram:

closed	$\Rightarrow$	g-closed	$\Rightarrow$	πg-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
α-closed	$\Rightarrow$	αg-closed	$\Rightarrow$	πgα-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
p-closed	$\Rightarrow$	gp-closed	$\Rightarrow$	πgp-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
b-closed	$\Rightarrow$	gb-closed	$\Rightarrow$	πgb-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
β-closed	$\Rightarrow$	gβ-closed	$\Rightarrow$	πgβ-closed

and

closed	$\Rightarrow$	g-closed	$\Rightarrow$	πg-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
α-closed	$\Rightarrow$	αg-closed	$\Rightarrow$	πgα-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
s-closed	$\Rightarrow$	gs-closed	$\Rightarrow$	πgs-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
b-closed	$\Rightarrow$	gb-closed	$\Rightarrow$	πgb-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
β-closed	$\Rightarrow$	gβ-closed	$\Rightarrow$	πgβ-closed

Where none of the above implications is reversible as can be seen from the following examples:

2.3 Example. Let X = {a, b, c, d} and $\Im$ = {∅, {a}, {b, c}, {a, b, c}, X}. Then A = {a} is gb-closed as well as gβ-closed but not closed.

2.4 Example. Let X = {a, b, c, d} and $\Im$ = {∅, {b}, {d}, {b, d}, X}. Then A = {a, b, d} is gb–closed as well as gβ-closed but it is not closed.

2.5 Example. Let X = {a, b, c, d} and $\Im$ = {∅, {b, d}, {a, b, d}, {b, c, d}, X}. Then A = {a, b} is gb–closed as well as πgb-closed but it is not closed.

2.6 Example Let X = {a, b, c} and τ = {ϕ, {b}, {c}, {b, c}, X}. Then the subset A = {b} is g-closed as well as gb-cloed but not closed.

2.7 Example. Let X = {a, b, c,} and τ = {ϕ, {a}, X}. Then the subset A = {a, b} is g-closed as well as gb-closed but not closed.

2.8 Example Let X = {a, b, c} and τ = {ϕ, {a}, {a, b}, X}. Then the subset A = {a, c} is g-closed as well gb-closed but not closed.

2.9 Example. Let X = {a, b, c, d, e} and ℑ = {∅, {a, b}, {b, d}, {a, b, c, d}, X}. Then A = {a, e} is πg–closed as well as πgα-closed but it is not closed.

2.10 Example. Let X = {a, b, c, d} and ℑ = {∅, {a}, {d}, {a, d}, {c, d}, {a, c, d}, X}. Then A = {c} is πgα–closed as well as πgp-closed but it is not closed.

2.11 Example. Let X = {a, b, c, d} and ℑ = {∅, {a}, {b}, {a, b}, {a, b, d}, X}. Then A = {a} is πgs–closed as well as πgb-closed but it is not closed.

2.12 Example. Let X = {a, b, c, d} and ℑ = {∅, {a}, {d}, {a, d}, {c, d}, {a, c, d}, X}. Then A = {c} is πgp–closed as well as πgβ-closed but it is not closed.

3. πgb-normal spaces

In this section, we introduce the notion of πgb–normal space and study some property of it. First, we begin with the following definitions and examples.

3.1 Definition. A space X is said to be **πgb–normal** (resp. **πgp–normal** [38], **πgβ–normal [10]**) if for every pair of disjoint πgb–closed (resp. πgp-closed, πgβ-closed) subsets H and K of X, there exist disjoint b–open (resp. p-open, β-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.2 Definition. A space X is said to be **γ–normal** [8] (resp. **p-normal** [24], **β-normal** [17]) if for every pair of disjoint closed subsets A, B of X, there exist disjoint γ–open (resp. p-open, β-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.3 Definition. A space X is said to be **πγ-normal** [11] (resp. **πp-normal** [36], **πβ–normal** [29]) if for every pair of disjoint closed subsets A, B of X, one of which is π–closed, there exist disjoint γ–open (resp. p-open, β-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.4 Definition. A space X is said to be **almost normal** [32] (resp. **almost p-normal** [21], **almost γ–normal** [12], **almost β-normal** [28]) if for any two disjoint closed subsets A and B of X, one of which is regularly closed, there exist disjoint open (resp. p-open, γ-open, β-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.5 Definition. A space X is said to be **softly normal** [31] (resp. **softly p-normal** [13], **softly γ–normal** [13], **softly β-normal**) if for any two disjoint subsets A and B of X, one of which is π–closed and other is regularly closed, there exist disjoint open (resp. p-open, γ-open, β-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.6 Definition. A space X is said to be **quasi-normal** [**39**] (resp. **quasi p–normal** [**37**], **quasi γ-normal** [**11**], **quasi β-normal** [**30**]) if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open (resp. p–open, γ-open, β-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.7 Definition. A space X is said to be **mildly-normal** [33] (resp. **mildly p–normal** [**21**], **mildly γ-normal** [**12**], **mildly β-normal** [**30**]) if for every pair of disjoint regularly closed subsets A, B of X, there exist disjoint open (resp. p–open, γ-open, β-open) sets U, V of X such that A ⊂ U and B ⊂ V.

Clearly, from above definitions, we have the following diagram:

	normal ⇒	π-normality ⇒	almost normal ⇒	softly normal ⇒	mildly normal
	⇓	⇓	⇓	⇓	⇓
πgp–normal ⇒	p-normal ⇒	πp-normal ⇒	almost p-normal ⇒	softly p-normal ⇒	mildly p-normal
⇓	⇓	⇓	⇓	⇓	⇓
πgb –normal ⇒	γ-normal ⇒	πγ-normal ⇒	almost γ-normal ⇒	softly γ-normal ⇒	mildly γ-normal
⇓	⇓	⇓	⇓	⇓	⇓
πgβ –normal ⇒	β-normal ⇒	πβ-normal ⇒	almost β-normal ⇒	softly β-normal ⇒	mildly β-normal

and

	normal ⇒	π-normal ⇒	quasi normal ⇒	softly normal ⇒	mildly normal
	⇓	⇓	⇓	⇓	⇓
πgp–normal ⇒	p-normal ⇒	πp-normal ⇒	quasi p-normal ⇒	softly p-normal ⇒	mildly p-normal
⇓	⇓	⇓	⇓	⇓	⇓
πgb–normal ⇒	γ-normal ⇒	πγ-normal ⇒	quasi γ-normal ⇒	softly γ-normal ⇒	mildly γ-normal
⇓	⇓	⇓	⇓	⇓	⇓
πgβ–normal ⇒	β-normal ⇒	πβ-normal ⇒	quasi β-normal ⇒	softly β-normal ⇒	mildly β-normal

Where none of the above implications is reversible as can be seen from the following examples:

3.8 Example. We consider the topology τ = {∅, {b, d}, {a, b, d}, {b, c, d}, X} on the set X = {a, b, c, d}. Then, the space X is p-normal as well as β–normal. But it is neither πgp–normal nor πgβ–normal.

3.9 Example. Let X = {a, b, c, d} and τ = {∅, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}, X}. Then the space X is β–normal as well as πβ–normal but it is not p-normal.

3.10 Example. Let X = {a, b, c, d} and τ = {∅, {a}, {c}, {a, c}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Also U = {a, b} and V = {c, d} are β–open sets such that $A \subset U$ and $B \subset V$. Hence X is β–normal as well as πβ-normal.

3.11 Example. Let X = {a, b, c, d} and τ = {∅, {a}, {b, d}, {a, b, d}, {b, c, d}, X}. Then, the space X is β–normal.

3.12 Example. Let X = {a, b, c} and τ = {∅, {a}, {a, b}, {a, c}, X}. Then the space X is πp-normal as well as πγ–normal but not p–normal.

3.13 Example. Let X = {a, b, c} and τ = {∅, {a}, {b}, {a, b}, X}. Then the space X is p-normal as well as πp–normal.

3.14 Example. Let X = {a, b, c, d} and τ = {∅, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Then the space X is γ-normal as well as πγ–normal but not p–normal.

3.15 Example. Let X = {a, b, c, d, e} and τ = {∅, {e}, {a, b}, {c, d}, {a, b, e}, {c, d, e}, {a, b, c, d}, X}. The pair of disjoint π-closed subsets of X are A = {a, b} and B = {c, d}. Also U = {a, b, e} and V = {c, d} are γ–open sets such that $A \subset U$ and $B \subset V$. Hence X is quasi γ–normal but not quasi-normal, since U and V are not open sets.

3.16 Theorem. For a topological space X, the following are equivalent:

(a) X is πgb-normal.

(b) For every pair of disjoint πgb-open subsets U and V of X whose union is X, there exist b-closed subsets G and H of X such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

(c) For every πgb-closed set A and every πgb-open set B in X such that $A \subset B$, there exists a b-open subset V of X such that $A \subset V \subset bcl(V) \subset B$.

(d) For every pair of disjoint πgb-closed subsets A and B of X, there exists a b-open subset V of X such that $A \subset V$ and $bcl(V) \cap B = \varnothing$.

(e) For every pair of disjoint πgb-closed subsets A and B of X, there exist b-open subsets U and V of X such that $A \subset U$, $B \subset V$ and $bcl(U) \cap bcl(V) = \varnothing$.

Proof. (a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (d), (d) ⇒ (e) and (e) ⇒ (a).

(a) ⇒ (b). Let U and V be any πgb–open subsets of a πgb–normal space X such that $U \cup V = X$. Then, $X \setminus U$ and $X \setminus V$ are disjoint πgb–closed subsets of X. By πgb–normality of X, there exist disjoint b–open subsets U_1 and V_1 of X such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $G = X \setminus U_1$ and $H = X \setminus V_1$. Then, G and H are b–closed subsets in X such that $G \cup H = X$.

(b) ⇒ (c). Let A be a πgb–closed and B is πgb–open subsets of X such that $A \subset B$. Then, $A \cap (X \setminus B) = \varnothing$. Thus, $X \setminus A$ and B are πgb–open subsets of X such that $(X \setminus A) \cup B = X$. By the Part (b), there exist b–

closed subsets G and H of X such that $G \subset (X \setminus A)$, $H \subset B$ and $G \cup H = X$. Thus, we obtain that $A \subset (X \setminus G) \subset H \subset B$. Let $V = X \setminus G$. Then V is b–open subset of X and $bcl(V) \subset H$ as H is b–closed set in X. Therefore, $A \subset V \subset bcl(V) \subset B$.

(c) ⇒ (d). Let A and B be disjoint πgb-closed subset of X. Then $A \subset X \setminus B$, where $X \setminus B$ is πgb-open. By the part (c), there exists a b-open subset U of X such that $A \subset U \subset bc1(U) \subset X \setminus B$. Thus, $bc1(U) \cap B = \emptyset$.

(d) ⇒ (e). Let A and B be any disjoint πgb-closed subset of X. Then by the part (d), there exists a b-open set U containing A such that $bcl(U) \cap B = \emptyset$. Since bcl(U) is b-closed, then it is πgb-closed. Thus bcl(U) and B are disjoint πgb-closed subsets of X. Again by the part (d), there exists a b-open set V in X such that $B \subset V$ and $bcl(U) \cap bcl(V) = \emptyset$.

(e) ⇒ (a). Let A and B be any disjoint πgb-closed subsets of X. Then by the part (e), there exist b-open sets U and V such that $A \subset U$, $B \subset V$ and $bcl(U) \cap bcl(V) = \emptyset$. Therefore, we obtain that $U \cap V = \emptyset$ and hence X is πgb-normal.

2.17 Definition. A subset A of a space X is said to be a **b–neighbourhood** [4] of x if there exists a b–open set U such that $x \in U \subset A$.

3.18 Definition. A function $f : X \to Y$ is said to be

(1) **regular open** [**26**] if f(U) is regular open in Y for every open set U in X.

(2) **π-continuous** [7] if $f^{-1}(F)$ is π-closed in X for each closed set F in Y.

(3) **strongly-b-closed** [**8**] (resp. **strongly b–open** [**8**]) f(F) is b-closed (resp. b-open) set in Y for every b–closed (resp. b–open) set F in X.

(4) **πgb-continuous** [**34**] if $f^{-1}(F)$ is πgb-closed in X for every closed set F in Y.

(5) **πgb-irresolute [34]** if $f^{-1}(F)$ is b-open in X for every b-open set F in Y.

(6) **b-irresolute [8]** if $f^{-1}(V)$ is b-open in X for every b-open set V in Y.

(7) **almost b–irresolute** [**8**] if for each $x \in X$ and b–neighbourhood V of f(x) in Y, $bcl(f^{-1}(V))$ is neighbourhood of x in X.

3.19 Lemma.

(a) The image of b-open subset under an open continuous function is b-open.

(b) The inverse image of b-open (resp. b-closed) subset under an open continuous function is b-open (resp. b-closed) subset.

3.20 Lemma [38]. The image of a regular open subset under open and closed continuous function is regular open subset.

3.21 Proposition [**38**]. The image of a π-open subset under open and closed continuous function is π-open set.

3.22 Proposition. If $f : X \to Y$ be an open and closed continuous bijection function and be a πgb-closed set in Y, then $f^{-1}(A)$ is πgb–closed set in X.

Proof. Let A be a πgb–closed subset of Y and U be any π–open subset of X such that $f^{-1}(A) \subset U$. Then by the **Proposition 3.21**, we have f (U) is π–open subset of Y such that $A \subset f(U)$. Since A is πgb-closed subset of Y and f(U) is π–open set in Y, thus $bcl(A) \subset U$. By the **Lemma 3.19**, we obtain that $f^{-1}(A) \subset f^{-1}(bcl(A)) \subset U$, where $f^{-1}(bcl(A))$ is b-closed in X. This implies that $bcl(f^{-1}(A)) \subset U$. Therefore, $f^{-1}(A)$ is πgb-closed set in X.

3.23 Theorem. πgb-normality is a topological property.

Proof. Let X be a πgb-normal space and $f : X \rightarrow Y$ be an open and closed bijection continuous function. We need to show that Y is πgb-normal. Let A and B be any disjoint πgb-closed subsets of Y. Then by the **Proposition 3.22**, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint of πgb-closed subsets of X. By πgb-normality of X, there exist b-open subsets U and V of X such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \varnothing$. By assumption, we have $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \varnothing$. By the **Lemma 3.19**, f(U) and f(V) are disjoint b-open subsets of Y such that $A \subset f(U)$ and $B \subset f(V)$. Hence, Y is πgb-normal.

4. πgb-normality in subspaces

4.1 Lemma. If M be an open subspace of a space X and $A \subset M$, then $bcl_M(A) = bcl_X(A) \cap M$.

4.2 Lemma [38].If M be an open subspace of a space X and $A \subset M$, then $int_M(cl_M(A) = int_X(cl_X(A) \cap M$.

4.3 Lemma [38]. If M be a π-open subspace of a space X and U be a π–open subset of X, then $U \cap M$ is π–open set in M.

4.4 Lemma. If A is both π–open and πgb-closed subset of a space X, then A is b-closed set in X.

Proof. Since A is πgb–closed and π–open subset of X and since $A \subset A$, then $bcl(A) \subset A$. But $A \subset bcl(A)$. Thus, $A = bcl(A)$. Hence, A is b–closed set in X.

4.5 Corollary. If A is both π–open and πgb-closed subset of a space X, then A is regular closed set in X.

4.6 Theorem. Let M be a π-open subspace of a space X and $A \subset M$. If M is πgb-closed set in X and A is πgb-closed set in M. Then A is πgb-closed set in X.

Proof. Suppose that M is πgb-closed set in X and A is πgb-closed set in M. Let U be any π–open set in X such that $A \subset U$. Then by **Lemma 4.3,** we have $A \subset M \cap U$, where $M \cap U$ is π–open set in M. Since A is πgb–closed in M, thus $bcl_M(A) \subset M \cap U$. The by the **Lemma 4.1**, $bcl_X(A) \cap M \subset M \cap U$. By the **Lemma 4.4**, we obtain that $\beta cl_X(M) = M$. Thus, $bcl_X(A) \subset bcl_X(M) = M$. So, $bcl_X(A) \cap M = bcl_X(A)$. Hence, $bcl_X(M) \subset U \cap M$. Thus, $bcl_X(A) \subset U$. Therefore, A is πgb–closed set in X.

4.7 Lemma. Let M be a closed domain subspace of a space X. If U is b–open set in X, then $U \cap M$ is b–open set in M.

4.8 Theorem. A πgb–closed and π–open subspace of a πgb–normal space is πgb–normal.

Proof. Let M be a πgb–closed and π–open subspace of a πgb–normal space X. Let A and B be any disjoint πgb–closed subsets of M. Then by **Theorem 4.6**, we have A and B are disjoint πgb–closed sets in X. By πgb–normality of X, there exist b–open subsets U and V of X such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$. By the **Corollary 4.5** and **Lemma 4.7**, we obtain that $U \cap M$ and $V \cap M$ are disjoint b–open sets in M such that $A \subset U \cap M$ and $B \subset V \cap M$. Hence, M is πgb–normal subspace of πgb–normal space X.

5. Preservation theorems for πgb-normal spaces

5.1 Definition. A function $f : X \to Y$ is said to be **π-irresolute** [**5**] if $f^{-1}(F)$ is π-closed in X for every π-closed set F in Y.

5.2 Theorem. If $f : X \to Y$ is π-irresolute, strongly b-closed and A is a πgb-closed subset of X, then f(A) is πgb-closed subset of Y.

Proof. Let A be a πgb-closed subset of X and U be any π-open set of Y such that $f(A) \subset U$. Then, $A \subset f^{-1}(U)$. Since f is π-irresolute function, then $f^{-1}(U)$ is π-open in X. Since A is πgb-closed set in X and $A \subset f^{-1}(U)$, then $bcl_X(A) \subset f^{-1}(U)$. This implies that $f(bcl_X(A)) \subset U$. Since f is pre b-closed and $bcl_X(A)$ is b–closed set in X, then $f(bcl_X(A))$ is b–closed in Y. Thus, we have $bcl_Y(f(A)) \subset U$. Hence, f(A) is πgb–closed subset of Y.

5.3 Corollary. If $f : X \to Y$ is π-continuous, strongly b-closed and A is a πgb-closed subset of X, then f(A) is πgb-closed subset of Y.

5.4 Theorem. If $f : X \to Y$ is π-irresolute, strongly b-closed and b-irresolute injection function from a space X to a πgb–normal Y, then X is πgb-normal.

Proof. Let A and B be any two disjoint πgb-closed subsets of X. By the **Theorem 5.2,** f(A) and f(B) are disjoint πgb-closed subsets of Y. By πgb–normality of Y, there exist disjoint b-open subsets U and V of Y such that $f(A) \subset U$, $f(B) \subset V$ and $U \cap V = \varnothing$. Since f is b-irresolute injection function, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint b-open sets in X such that $A \subset f^{-1}(U)$ and $B \subset f^{-1}(V)$. Hence X is πgb-normal.

5.5 Corollary. If $f : X \to Y$ is π-continuous, strongly b-closed and b-irresolute injection function from a space X to a πgb–normal Y, then X is πgb-normal.

5.6 Lemma. If the bijection function $f : X \to Y$ is b–continuous and regular open, then f is πgb–irresolute.

5.7 Theorem. If $f : X \to Y$ is πgb-irresolute, strongly b-closed bijection function from a πgb–normal space X to a space Y, then Y is πgb–normal.

Proof. Let A and B be any two disjoint πgb-closed subsets of Y. Since f is πgb–irresolute, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint πgb-closed subsets of X. By πgb–normality of X, there exist disjoint b-open sets U and V in X such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \varnothing$. Since f is pre b–open and bijection function, we have f(U) and f(V) are disjoint b-open sets in Y such that $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \varnothing$. Therefore, X is πgb-normal.

5.8 Corollary. If $f : X \to Y$ is b-continuous, regular open and strongly b-open bijection function from a πgb–normal space X to a space Y, then Y is πgb–normal.

5.9 Theorem. If $f : X \to Y$ is a strongly b–open, πgb–irresolute and almost b–irresolute surjection function from a πgb–normal space X onto a space Y, then Y is πgb–normal.

Proof. Let A be a πgb–closed subset of Y and B be a πgb–open subset of Y such that $A \subset B$. Since f is πgb-irresolute, we obtain that $f^{-1}(A)$ is πgb–closed in X and $f^{-1}(B)$ is πgb–open in X such that $f^{-1}(A) \subset f^{-1}(B)$. Since X is πgb–normal, then by the Part (c) of the **Theorem 3.16**, there exists a b–open set U of X such that $f^{-1}(A) \subset U \subset bcl_X(U) \subset f^{-1}(B)$. Then, $f(f^{-1}(A)) \subset f(U) \subset f(bcl_X((U)) \subset f(f^{-1}(B)$. Since f is pre b–open, almost b–irresolute surjection, we obtain that $A \subset f(U) \subset bcl_Y(f(U)) \subset B$ and f(U) is b–open set in Y. Hence by the **Theorem 3.16**, we have Y is πgb–normal.

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