GENERALIZATION OF SOMEWHAT CONTINUOUS FUNCTIONS

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Abstract: The aim of this paper is to introduce and study somewhat sgα-continuous functions on generalized topological space.
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1. INTRODUCTION

In 1963, Levine [2] initiated the study of so-called semi-open sets. The notion has been studied extensively in recent years by many topologists. The notion somewhat continuous functions plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map f: X→Y when both f and f^{-1} are continuous. As generalization of closed sets, sgα-closed sets were introduced and studied by Rajesh and Krsteska [5].introduced the concept of generalized closed maps in topological spaces. In this paper, to introduce and study somewhat sgα-continuous functions on topological space.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A, respectively. If $A \subseteq B \subseteq X$, then $cl_B(A)$ and $int_B(A)$ denote the closure of A relative to B and an interior of A relative to B.

We recall the following definitions, which are useful in the sequel.

Definition 2.1: A subset A of a space (X, τ) is called

(i). a semi-open set [3] if A⊆cl(int(A)) and

(ii).an α-open set [5] if A⊆int(cl(int(A))).

The complement of α-closed set is called α-open. The α-closure [4] of a subset A of X, denoted by $\alpha cl_X(A)$ (briefly αcl(A)) is defined to be the intersection of all α-closed sets containing A.

Definition 2.2: A subset A of a topological space (X, τ) is called a semi-generalized α-closed (briefly sg-α-closed) set [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of sg-α-closed sets is called sg-α-open.

2. SOMEWHAT SGα-CONTINUOUS FUNCTIONS

Definition 2.1. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat sgα-continuous if for $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$, there exists a open set V in X such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.

Definition 2.2. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be sgα-continuous if [1] if for $U \in \sigma$, there exists an open set V in X such that $V \subset f^{-1}(U)$.

It is clear that every sgα-continuous function is somewhat sgα-continuous but the converse is not true as shown by the following examples.

Example 2.3. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{a, c\}\}$ and $\sigma = \{\emptyset, X, \{a, b\}\}$. Then the identity function f: (X, τ) → (Y, σ) is somewhat sgα-continuous but not sgα-continuous.

Definition 2.4. A subset M of a topological space is said to be sgα-dense in X if there is no proper sgα-closed set C in X such that $M \subset C \subset X$

Theorem 2.5. For a surjective function f: (X, τ) → (Y, σ), the following statements are equivalent:

(i) f is somewhat sgα-continuous

(ii) If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper sgα-closed subset D of X such that $D \supset f^{-1}(C)$.

(iii) If A is a open subset of Y such that $f^{-1}(A) \neq X$, then there is a proper sgα-open subset B of X such that $f^{-1}(A) = B$;

(iv) If M is a sgα-dense subset of X, then f(M) is dense subset of Y.

Proof. (i) ⇒ (ii): Let C be a open subset of Y such that $f^{-1}(C) \neq X$. Then $Y \backslash C$ is open set in Y such that $f^{-1}(Y \backslash C) = X \backslash f^{-1}(C) \neq \emptyset$. By (i), there exists a sgα-open set V in X such that $V \neq \emptyset$ and $V \subset f^{-1}(Y \backslash C) = X \backslash f^{-1}(C)$. This means that $X \backslash V \supset f^{-1}(C)$ and $X \backslash V = D$ is a proper sgα-closed set in X.

(ii) ⇒ (i): If U is open in Y and $f^{-1}(U) \neq \emptyset$. Then $Y \backslash U$ is closed and $f^{-1}(Y \backslash U) = X \backslash f^{-1}(U) \neq X$. By (ii), there exists a proper sgα-closed set D such that $D \supset f^{-1}(Y \backslash U)$. This implies $X \backslash D \subset f^{-1}(U)$and $X \backslash D$ is sgα-open and $X \backslash D = \emptyset$.

(ii) ⇔(iii): Clear.

(ii) ⇔(iv): Let M be a sgα-dense set in X. Suppose that f(M) is not dense in Y. Then there exists a proper closed set C in Y such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. By (ii), there exists a proper sgα-closed set D such that $M \subset f^{-1}(C) \subset D \subset X$. This is a contradiction to the fact that M is sgα-dense in X.

(iii) ⇔(ii): Suppose (ii) is not true. This means that there exists a closed set C in Y such that $f^{-1}(C) \neq X$ but there is no proper sgα-closed set D in X such that $f^{-1}(C) \subset D$. This means that $f^{-1}(C)$ is sgα-dense in X. But by (iii), $f\left(f^{-1}(C)\right) = C$ must be dense in Y, which is a contradiction to the choice of C.

Definition 2.6. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be somewhat sgα-open provided that if for every sgα-open set U of X and $U \neq \emptyset$, then there exists a open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.

Definition 2.7. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be somewhat sgα-open provided that if U is sgα-open in X and $U \neq \emptyset$, then there exists a open set V in Y such that $V \subset f(U)$.

It is clear that every open function is somewhat sgα-open but the converse is not true as the following example shows.

Example 2.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. Then the identity function $f: (X, \tau) \to (Y, \sigma)$is somewhat sgα-open but not open.

Proposition 2.9. For a bijection function $f: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(i) f is somewhat sgα-open.

(ii) If C is a sgα-closed subset of X, such that $f(C) \neq Y$, then there is a closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$.

Proof. (i) ⇒ (ii): Let C be a sgα-closed subset of X such that $f(C) \neq Y$. Then $X \backslash C$ is sgα-open in X and $X \backslash C \neq\neq \emptyset$. Since f is somewhat sgα-open, there exists a open set $V \neq \emptyset$ in Y such that $V \subset f(X \backslash C)$. Put $D = Y \backslash V$. Clearly D is closed in Y and we claim $D \neq Y$. If $D = Y$ then $V = \emptyset$, which is a contradiction. Since $V \subset f(X \backslash C)$, $D = Y \backslash V \supset \left(Y \backslash f(X \backslash C)\right) = f(C)$.

(ii) ⇒ (i): Let U be any nonempty sgα-open subset of X. Then $C = X \backslash U$ is a sgα-closed set in X and $f(X \backslash U) = f(C) = Y \backslash f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is a closed set D of Y such that $D \neq Y$ and $f(C) \subset D$. Clearly $V = Y \backslash D$ is a open set and $V \neq \emptyset$. Also, $V = Y \backslash D \subset Y \backslash f(C) = Y \backslash f(X \backslash U) = f(U)$.

Proposition 2.10. For a function $f: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(i) f is somewhat sgα-open.

(ii) If A is a dense subset of Y, then $f^{-1}(A)$ is a sgα-dense subset of X.

Proof. (i) ⇒ (ii): Suppose A is a dense set in Y . We want to show that $f^{-1}(A)$ is a sgα-dense subset of X. Suppose not, then there exists a sgα-closed set B in X such that $f^{-1}(A) \subset B \subset X$. Since f is somewhat sgα-open and X\B is sgα-open, there exists a nonempty sgα-open set C in Y such that $C \subset f(X\backslash B)$. Therefore, $C \subset f(X\backslash B) \subset f(f^{-1}(Y\backslash A)) \subset Y\backslash A$. That is, $A \subset Y\backslash C \subset Y$. Now, Y\C is a closed set and $A \subset Y\backslash C \subset Y$. This implies that A is not a dense set in Y, which is a contradiction. Therefore, $f^{-1}(A)$ must be a sgα-dense set in X.

(ii) ⇒ (i): Suppose A is a nonempty open subset of X. We want to show that $int(f(A)) \neq \emptyset$. Suppose $int(f(A)) \neq \emptyset$. Then $cl(Y\backslash f(A)) = Y$. Therefore, by (ii), $f^{-1}(Y\backslash f(A))$ is sgα-dense in X. But $f^{-1}(Y\backslash f(A)) \subseteq X\backslash A$. Now X\A is sgα-closed. Therefore, $f^{-1}(Y\backslash f(A)) \subseteq X\backslash A$ gives X = sgαcl $(f^{-1}(Y\backslash f(A))) \subseteq X\backslash A$. This implies that $A = \emptyset$, which is contrary to $A = \emptyset$. Therefore, $int(f(A)) = \emptyset$. This proves that f is somewhat sgα-open.

Definition 2.11. A topological space (X, τ) is said to be sgα-resolvable if there exists a sgα-dense set A in (X, τ) such that A^c is also sgα-dense in (X, τ). Otherwise, (X, τ) is called sgα-irresolvable.

Theorem 2.12. For a topological space (X, τ), the following statements are equivalent:

(i) (X, τ) is sgα-resolvable;

(ii) (X, τ) has a pair of sgα-dense set A and B such that $A \subseteq B^c$.

Proof. $(i) \Rightarrow (ii)$: Suppose that (X, τ) is sgα-resolvable. There exists a sgα-dense set A such that $X\backslash A$ is sgα-dense. Set $B = X\backslash A$, then we have $A = X\backslash B$.

$(ii) \Rightarrow (i)$: Suppose that the statement (ii) holds. Let (X, τ) be sgα-irresolvable. Then $X\backslash B$ is not sgα-dense and sgα-cl$(A) \subset sg\alpha - cl(X\backslash B) \neq X$. Hence A is not sgα-dense. This contradicts the assumption.

Theorem 2.13. For a topological space (X, τ), the following statements are equivalent:

(i) (X, τ) is sgα-irresolvable;

(ii) For any sgα-dense set A in $X, sg\alpha - int(A) \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$: Let A be any sgα-dense set of X. Then sgα-cl$(X\backslash A) \neq X$; hence $sg\alpha - int(A) \neq \emptyset$.

$(ii) \Rightarrow (i)$: Suppose that (X, τ) is a μ-resolvable space. Then there exists a sgα-dense set A in (X, μ) such that A^c is also sgα-dense in X. It follows that $sg\alpha - int(A) \neq \emptyset$, which is a contradiction; hence (X, μ) is sgα-irresolvable.

Theorem 2.14. If $\bigcup_{I=1}^{n} A_i = X$, where A_i's are subsets of X such that sgα-int$(A) \neq \emptyset$, then (X, τ) is a sgα-irresolvable.

Proof. By hypothesis, we have $\bigcap_{i=1}^{n}(X\backslash A_i) = \emptyset$. Then, there must be at least two nonempty disjoint subsets A_i^c and A_j^c in X. That is $A_i^c \cup A_j^c \subset X$. Then $A_i^c = A_j$; hence sgα-cl$(A_j) = X$. Also sgα-int$(A_j) = \emptyset$ implies that sgα-c$(X\backslash A_j) = X$.. Therefore, (X,τ) has a sgα-dense set A_j such that sgα-cl$(A_j) = X$. Hence (X,μ) is a sgα-irresolvable.

Theorem 2.15. If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a somewhat sgα-open function and int$(A) = \emptyset$ for a nonempty set A in Y , then sgα-int$(f^{-1}(A)) = \emptyset$.

Proof. Let A be a nonempty set in Y such that int$(A) = \emptyset$. Then cl$(Y\backslash A) = Y$. Since f is somewhat sgα-open and $Y\backslash A$ is dense in Y, by Proposition 2.10 $f^{-1}(Y\backslash A)$ is sgα-dense in X. Then, sgα-cl$(X\backslash f^{-1}(A)) = X$; hence sgα-int$(f^{-1}(A)) = \emptyset$.

Theorem 2.16. Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be a somewhat sgα-open function. If X is sgα-irresolvable, then Y is irresolvable.

Proof. Let A be a nonempty set in Y such that cl$(A) = Y$. We show that int$(A) = \emptyset$. Suppose not, then cl$(Y\backslash A) = Y$. Since f is somewhat sgα-open and $Y\backslash A$ is dense in Y, we have by Proposition 2.10 $f^{-1}(Y\backslash A)$ is dense in X. Then sgα-int$(f^{-1}(A)) = \emptyset$ .Now, since A is dense in Y, $f^{-1}(A)$ is sgα-dense in X. Therefore, for the sgα-dense set $f^{-1}(A)$, we have sgα-int$(f^{-1}(A)) = \emptyset$, which is a contradiction to Theorem 2.13. Hence we must have int$(A) \neq \emptyset$ for all dense sets A in Y. Hence by Theorem 2.13, Y is irresolvable.

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