SOME GENERALIZATION OF KANNAN'S FIXED POINT THEOREM IN B-METRIC SPACES WITH A REFLEXIVE DIGRAPH

Anuradha
Research Scholar
Department of Mathematics,
Mahrashi Dayanand University, Rohtak, India

Abstract: The purpose of this paper is to establish some fixed point theorem in the frame work of b- metric space with a reflexive digraph. These results substantially extend and generalize the Kannan fixed point theorem.

MSC: Primary 47H10, Secondary 54H25
Keywords: fixed point, b-metric space, reflexive digraph

1. Introduction and Preliminaries:

The area of the fixed point theory has wide number of application in applied mathematics and sciences. Recently, the common fixed points of mappings satisfying certain contractive conditions has been studied extensively by many authors. French mathematician Maurice Frechet in 1906 introduced the concept of metric space. After the work of French, several generalization of metric space came out by several mathematicians and one of the generalization is b- metric space that one introduced by Bakhtin [1] in 1989 and also generalized the famous Banach contraction principle in metric spaces to b-metric spaces. Since then, various number of articles came out in the improvement of fixed point theory in b-metric spaces.

Let X be a nonempty set. A mapping d: X×X→R is called b-metric on X if the following properties are satisfied:

1. $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y \in X$.

Then (X, d) is known as b-metric space.

Theorem [7]: (Kannan) Let (Q, d) be a complete metric space and let T: Q→Q be a mapping satisfying

$$d(Tx, Ty) \leq K [d(x, Tx)+ d(y, Ty)]$$

for all $x, y \in Q$ and $K < \frac{1}{2}$. Then T has a unique fixed point $z \in Q$ and for any $x \in Q$ the sequence of iterates converges to fixed point z.

Over the past few decades, there have been a lot of activity about combing fixed point theory and another branches in mathematics such as differential equations, geometry, optimization and algebraic topology. The study of fixed point theory endowed with a graph occupies a prominent place in many aspects. In 2005, Echenique [4] studied fixed point theory by using graphs. Espinola and Krik[5] applied fixed point results in graph theory. Graph theory have wide number of applications in many fields. A very interesting approach in the theory of fixed point with graph was recently given by Jachymski [6]. In recent investigation, the study of fixed point theory endowed with a graph plays an important role in many fields. Using this interesting idea, Jachymski studied the Banach contraction principle in metric spaces with a graph. His work extends and subsumes many recent results obtained on partially ordered metric spaces. Jachymski [6] uses the concept of graph instead of partial order in his paper. A lot of work in fixed point theory in various spaces endowed with a graph have been done by changing some conditions on mapping and spaces, by redefining some definitions on mappings like G-contraction, G-kannan, G-monotone non-expansive,φ -contraction, G- graphic contraction etc.

Our graph theory terminology and notations are standard and can be found in all graph theory books. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in N$) is a sequence $(x_i)_{i=0}^{i=n}$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E(\tilde{G})$ for $i = 0, 1, 2, \ldots n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected. Throughout this paper, we take (Q, d, G) be a b-metric space with coefficient $s \geq 1$ and reflexive digraph
G = (V, E), where V(G) = Q.

The purpose of this paper to prove some fixed point theorems in the context of b- metric space with a graph which generalize the well-known Kannan fixed point results in the literature.

2 Main Results

Let B be a non-empty subset of b- metric space (Q, d) with coefficient $s \geq 1$ and T: B→B a mapping then a sequence $\{x_n\}$ is said to be an approximating fixed point sequence of T if $d(x_n, Tx_n) \to 0$ $as\ n \to \infty$

Theorem 2.1 Let (Q, d, G) be a b- metric space with graph G, coefficient $s \geq 1$ and let S_1, S_2 are closed and compact subsets of X respectively. Let $T : S_1 \to S_2$ be a mapping such that there exist K< 1 with $Ks \neq 1$ satisfying

$$d(Tx, Ty) \leq K\,[d(x, Tx)+ d(y, Ty)] \qquad \forall\ x, y \in E(G) \qquad (1)$$

Then T has a unique fixed point if there exist an approximating fixed point sequence $\{x_n\}$ of T with $(z, x_n) \in E(G)$, if it converges to z.

Proof: Let $\{x_n\} \in S_1$ be an approximating fixed point sequence of T. Since $Tx_n \in S_2$, so by definition of S_2, there exist subsequence $Tx_{n_k} \to z \in S_2$ as $n \to \infty$. By assumption, that is, $\{x_n\}$ be an approximating fixed point sequence, it follows that $x_{n_k} \to z \in S_1$ then

$$\begin{aligned} d(Tz, z) &\leq s\,[d(Tz, Tx_{n_k})+ d(Tx_{n_k}, z)] \\ &\leq K\,s\,[d(z, Tz)+ d(x_{n_k}, Tx_{n_k})] + s\,d(Tx_{n_k}, z) \end{aligned}$$

Or we can write as

$$d(Tz, z) \leq \frac{Ks}{1-Ks}\, d(x_{n_k}, Tx_{n_k}) + \frac{s}{1-Ks}\, d(Tx_{n_k}, z) \to 0 \ \text{ as } n \to \infty$$

implies that Tz = z and hence it is the unique fixed point.

Consider now the situation in which T: Q→ Q is not necessarily a Kannan mapping, but T^P is a Kannan mapping for some $P \geq 2$.

Corollary 2.2. Let (Q, d, G) be a b- metric space with graph G and coefficient $s \geq 1$. Let $T : Q \to Q$ is a mapping such that for some positive integer $P \geq 2$, T^P is a mapping such that there exist $K< \frac{1}{2}$ with $Ks \neq 1$ satisfying

$$d(T^P x, T^P y) \leq K\,[d(x, T^P x)+ d(y, T^P y)] \qquad \forall\ x, y \in E(G) \qquad (2)$$

Then T has a unique fixed point if there exist an approximating fixed point sequence $\{x_n\}$ of T with $(z, x_n) \in E(G)$, if it converges to z.

Proof: It can be proved easily by [9] and Theorem 2.1

Theorem 2.2. Let (Q, d, G) be a b- metric space with graph G and coefficient $s \geq 1$. Let $T : Q \to Q$ be a mapping such that for some positive integer $P \geq 2$, there exist $K< \frac{1}{2}$ satisfying

$$d(T^P x, T^P y) \leq K\,[d(x, T x)+ d(y, T y)] \qquad \forall\ x, y \in E(G) \qquad (3)$$

If there is an $z \in Q$ such that $T^P z = z$ with $(T^i z, T^{i+1} z) \in E(G)$ for i= 0,1,2,…. then T has a unique fixed point. Furthermore, it is unique if z, w ∈ Q are two fixed points of T then $(z,w) \in E(G)$.

Proof: By assumption, there is an $z \in Q$ such that $T^P z = z$, then by (3)

$$d(z, Tz) = d(T^P z, T^{P+1} z) \leq K\,[d(z, T z)+ d(T z, T (T z))]$$

implies

$$d(z, Tz) \leq \frac{K}{1-K}\, d(T z, T^2 z).$$

Again by (3),

$$d(T z, T^2 z) = d(T^{P+1} z, T^{P+2} z) \leq K\,[\, d(Tz, T^2 z)+ d(T^2 z, T^3 z)\,]$$

that is,

$$d(T z, T^2 z) \leq \frac{K}{1-K}\, d(T^2 z, T^3 z)$$

Proceeding in this way, we obtain

$$d(T^{P-2} z, T^{P-1} z) \leq \frac{K}{1-K}\, d(T^{P-1} z, T^P z) \qquad (4)$$

and

$$d(z, Tz) \leq \left(\frac{K}{1-K}\right)^{P-1} d(T^{P-1} z, T^{P} z) \quad (5)$$

Now, $d(T^{P-1} z, T^{P} z) = d(T^{P+(P-1)} z, T^{P+P} z)$

$\leq K[d(T^{P-1} z, T^{P} z) + d(T^{P} z, T^{P+1} z)]$

Implies

$$d(T^{P-1} z, T^{P} z) \leq \frac{K}{1-K} d(z, Tz)$$

So, (5) becomes

$$d(z, Tz) \leq \left(\frac{K}{1-K}\right)^{P} d(z, Tz) \quad \text{as } K < \frac{1}{2}, \quad \text{so} \quad z = Tz.$$

Suppose that $z, w \in Q$ are two fixed points of T then $d(z, w) = d(T^{P} z, T^{P} w) \leq K [\, d(z, Tz) + d(w, Tw)] = 0$. Hence $z = w$.

References:

1. Bakhtin, I. A. 1989. The contraction mapping principle in almost metric space. Funct. Anal., Gos, Ped. Inst. Unianowsk, 30, 26–37.
2. Bojor, F. 2010. Fixed point of φ –contraction in metric spaces endowed with a graph. Mathematics and Computer Science Series, 37 (4), 85-92.
3. Bojor, F. 2012. Fixed points of Kannan mapping in metric spaces endowed with a graph, An St. uni. Ovidius Constanta, 20 (1), 31- 40
4. Echenique, F. 2005. A short and constructive proof of Tarski's fixed point theorem. Internat. J. Game Theory, 33, 215-218.
5. Espinola, R. and Krik, W.A. 2006. Fixed point theorems in R–trees with applications to graph theory. Topology Appl., 153, 1046-1055
6. Jachymski, J. 2007. The contraction principle for mappings on a metric space with a graph. American mathematical society, 136 (4), 1359-1373.
7. Kannan, R. 1968. Some results on fixed points. *Bull. Calcutta Math*. 60, 71-78.
8. Kirk, W.A. 1983. Fixed point theory for non-expansive mappings II. Contemporary mathematics 18, 121-140.