ON TOPOLOGICAL GAMES

[1]Kumari Julee & [2]Dr. Kamala Parhi
[1] Research Scholar, [2] Associate Professor
[1] Univ. Dept. of Math., T.M. Bhagalpur University, Bhagalpur, Bihar
[2] Dept. of Mathematics, Marwari College, Bhagalpur, T.M. Bhagalpur University, Bhagalpur, Bihar, India

Abstract : A topological game is an infinite game of perfect information played between two players on a topological space. The term topological game was first introduced by Claude Berge. Present paper is the study of topological game. Let X be a topological space and $I \subset P(X)$ s.t. I is closed w.r.t. union and I possesses heriditory property. Such I is defined as an ideal over X and an infinite positional game of pursuit and evasion on such I is played very well in this paper.

Key words: topological game, infinite positional game, possesses heriditory property, pursuit, evasion.

A topological game is an infinite game of perfect information played between two players on a topological space. The term topological game was first introduced by Claude Berge, [1][2][3] who defined the basic ideas and formalism in analogy with topological groups. A different meaning for topological game, the concept of 'topological properties defined by games', was introduced in the paper of R. Telgársky, [4] and later 'spaces defined by topological games'; [5] this approach is based on analogies with matrix games, differential games and statistical games, and defines and studies topological games within topology. After more than 35 years, the term "topological game" became widespread, and appeared in several hundreds of publications. The survey paper of Telgársky [6] emphasizes the origin of topological games from the Banach–Mazur game.

Let $G(I, X)$ be an infinite positional game of pursuit and evason over I where X is a topological space and $I \subset P(X)$ s.t.

(i) I is closed with respect to union;

(ii) I possesses heriditory property.

Such collection I is called an ideal over X. This game is played as follows:

There are two players - P (Pursuer) and E (Evader). They choose alternately consecutive terms of a sequence

$$\{E_n/n \in N, \text{ where } N = \{0, 1, 2, \ldots, n, \ldots\}\}$$

of subsets of X s.t. each player knows $I, E_0, E_1, \ldots, E_n$ when he is choosing E_{n+1}.

A sequence $\{E_n\}$ of subset of X is said to be a play of the game if for all $n \in N$ in the following holds:

(i) $E_0 = X$.

(ii) $E_1, E_3, E_5, \ldots, E_{2n+1}$ are the choice of P.

(iii) $E_1, E_3, E_5, \ldots, E_{2n+1} \in 1$

(iv) $E_2, E_4, E_6, \ldots, E_{2n+1}$ are the choice of E.

(v) $E_1, E_2 \subset E_0; E_3, E_4 \subset E_2; \ldots; E_{2n+1}, E_{2n+2} \subset E_{2n}$

(vi) $E_1 \cap E_2 = \phi, E_3 \cap E_4 = \phi, \ldots, E_{2n+1} \cap E_{2n+2} = \phi$.

If $\cap \{E2n\} = \phi$, then player P wins the play, otherwise Evader wins the play.

A finite sequence $\{E_m/m \le n\}$ is admissible for the game if the sequence $\{E_0, E_1, E_2, \ldots, \phi, \phi, \phi, \ldots, \phi\}$ is a play of the game.

For admissible sequence $\{E_0, E_1, E_2, \ldots, E_n\}$ and even n if

$$s : \{E_0, E_1, E_2, \ldots, E_n\} \to P(X)$$

and

$$s : (\{E_0, E_1, E_2, \ldots, E_n\}) = E_{n+1}$$

then s is a strategy for player P.

In case of odd n, s is said to be strategy for evader E.

A strategy s is said to be winning for player P in the game $G(I, X)$ if P wins each play of the game with the help of this s.

Similarly s is said to be winning for E if E wins each play of the game with the help of s.

We denote by $P(I, X)$, the set of all winning strategies of P in the game $G(I, X)$ and by $E(I, X)$, the set of all winning strategies of E in the game $G(I, X)$.

A topological space X is said to be I-like if the set of all winning strategies of player P is not empty i.e. if

$$P(I, X) \ne \phi.$$

Similarly, a space X is said to be anti I-like if the set of all winning strategies of player E is not empty. That is

$$E(I, X) \ne \phi.$$

The game $G(I, X)$ is said to be determined, if

$$P(I, X) \ne \phi.$$

or

$$E(I, X) \ne \phi$$

i.e. if X is I-like or X is anti I-like.

Thus we have the following properties:

If $I_1, I_2 \subset P(X)$, then

$$P_1 \quad I_1 \subset I_2 \Rightarrow P(I_1, X) \subset P(I_2, X)$$
$$\Rightarrow [P(I_1, X) \ne \phi$$
$$\Rightarrow P(I_2, X) \ne \phi];$$
$$P_2 \quad I_1 \subset I_2 \Rightarrow E(I_2, X) \subset E(I_1, X)$$

$$\Rightarrow [E(I_2, X) \neq \phi$$
$$\Rightarrow E(I_1, X) \neq \phi].$$

Theorem: 1. If the game $G(I, X)$ is determined in the favour of player P then the game $G(I, E)$ is also determined in favour of P for all $E \in P(X)$.

Proof. Let $s \subset P(1, X)$. We set

$$E_0 = X,$$
$$F_0 = E,$$
$$E_1 = s\{E_0\},$$
$$F_1 = F_0 \cap E_1.$$

We form $t \in P(I, E)$ such that

$$t\{F_0\} = F_1,$$

then

$$F_1 \in P(E) \cap I.$$

Let $F_2 \in P(E)$ with $F_1 \cap F_2 = \phi$.

We set

$$E_2 = F_2,$$
$$E_3 = s\{E_0, E_1, E_2\},$$
$$F_3 = E_3,$$

And $$t\{F_0, F_1, F_2\} = F_3.$$

Continuing in this manner, the plays $\{E_n\}$ of G (I, X) and $\{F_n\}$ of G (I, E) are obtained.

Now $E_{2n} = F_{2n}$ for all $n \geq 1$

$\Rightarrow \cap\{E_{2n}\} = \cap \{F_{2n}\}$

But $s \in P$ (I, X)

$\Rightarrow \cap\{E_{2n}\} = \phi.$

It follows that

$\cap\{E_{2n}\} = \phi$

$\Rightarrow t \in P$ (I, E).

Hence the theorem.

Theorem 2. If $E \in P(X)$ and $E(I, E) \neq \phi$, then $E(I, X) \neq \phi$.

Proof. Let $s \in E(I, X)$. To prove the theorem, it will be sufficient to define $t \in E(I, X)$. For set

$$E_0 = X,$$
$$F_0 = E,$$

and let $E_1 \in P(X) \cap I$.

Also set $F_1 = F \cap E_1$,

$$F_2 = s\{F_0, F_1\},$$
$$E_2 = F_2,$$

and $t\{E_0, E_1\} = E_2$.

Assume $E_3 \in P(X) \cap I$ with $E_0 \in E_2$.

Again set $F_3 = E_0$,

$$F_4 = s\{F_0, F_1, F_2, F_3\},$$
$$E_4 = F_4$$

and $$t\{E_0, E_1, E_2, E_3\} = E_4.$$

Continuing in this manner, the plays $\{E_n\}$ of $G(I, X)$ and $\{F_n\}$ of $G(I, E)$ are obtained such that

$$E_{2n} = F_{2n} \text{ for all } n \geq 1 \Rightarrow \cap\{E_{2n}\} = \cap\{F_{2n}\}.$$

But $$s \in E(I, E) \Rightarrow \cap(F2n) \neq \phi.$$

Hence $$\cap\{E2n\} \neq \phi \Rightarrow t \in E(I, X)\ E(I, X) \neq \phi.$$

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