ACTION OF SETS

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Abstract : The classical definition of group and its action is applicable to structures which are either Groups or Rings or Vector spaces etc. It finds its natural application in Physics, Chemistry (specially in Dynamics). Application of the above concept to other areas like Economics, Finance and in general to the areas which do not have the classical mathematic structures, needs extension of the classical definitions to general sets. In this paper we first recall the classical definition of group action which later is extended to sets. The inception of the concept of 'extension to sets' is established by incorporating sets like those of the integers and natural numbers which do not have a group structure. Hence in order to apply action of groups on these sets (non-group structured sets) we need to extend this concept to first define 'action of sets'. In this paper we try to find the action of Integers on Rational numbers, Real numbers or Complex numbers which necessitates an extension of the classical definition of group action to sets. The new definitions and the following elementary examples are applied to practical situations to accomplish our purpose/intention.

This paper contains an introductory section, a section giving a generalized version of definitions specified in the introduction with examples, some theorems extended to the new definitions and finally the conclusion section.

Sections comprising the paper are:

Section 1: Introduction and group definitions in various forms

Section 2: One-point and two-point Basic Action of a set

Section 3: One-point and two-point action on a set and on its subset

Section 4: Stabilizer interpreted under the basic action

Section 5: Orbit of set under the influence of action of set

Section 6: Extension of classical theorems on orbits and stabilizers

Section 7: Stability of action of group

SECTION 1: INTRODUCTION:

The action of a group on a set and other concepts associated with it are well-known. The concept of the action of a group on a set is a fundamental one, since groups find their application through their action. The structure of an action can be understood by means of its orbits and stabilizers. Our aim is to extend the definition of action-of-a-group-on-a-set to action of sets. For this purpose we recall the definition of a group and action of a group in various forms and then extend these to define action of sets. In this connection we define one-point and two-point basic actions and then define one-point and two-point actions on sets. We study orbits and stabilizers from this point of view.

It will be seen that the stabilizer of a point is a subgroup in the classical definition of action of a group but is not so in the general definition of action on sets. It can be observed that in the case of action of a group, the stabilizer of a single point is a subgroup and that of two or more points need not be. However, we examine some of the classical theorems in terms of these new definitions.

Notation: Throughout this paper, G denotes a group and X, Y denote sets.

Group definition 1.1:

For any set G, let f : G × G → G be a mapping. (G, f) is called a group if

(i) f(a, f(b, c)) = f(f(a, b), c)

(ii) ∃ a unique e ∈ G such that f(e, a) = f(a, e) = a ∀a ∈ G

(iii) For each a ∈ G ∃ a unique b ∈ G such that f(a, b)= f(b, a) = e

If we denote f(a, b) by a·b then the above conditions become

$$a\cdot (b\cdot c) = (a\cdot b)\cdot c$$
$$a\cdot e = e\cdot a = a$$
$$a\cdot b = b\cdot a = e$$

Group Definition 1.2:

We can also define a group through a mapping on G. Let f_g: G → G be a mapping for each g ∈ G such that

$$f_a\,(f_b(c)) = f_{f_a(b)}(c)$$
$$f_a\,(e) = f_e(a) = a$$
$$f_a\,(b) = f_b(a) = e$$

Then (G, f_g) / g ∈ G is called a group.

Using the above forms of group definitions, we proceed to define action of a group as below:

Definition 1.3: Action of a group (G, f) on a set:

A group (G, f) is said to act on a set X if ∃ a function ψ: G × X → X such that

(i) ψ(e, x) = x

(ii) $\psi(f(g_1, g_2), x) = \psi(g_1, \psi(g_2, x))$

Writing ψ(g, x) as gx we get

(i) ex = x

(ii) $f(g_1, g_2)x = (g_1, (g_2 x))$
$= g_1 (g_2 x)$

Note that gx ∈ X, hence g: X → X i.e., we have interpreted elements of G as mappings on X. This gives the clue to define action of a set on itself and on another set.

Definition 1.4: Group Homomorphism

Let (G, f) and (G^*, f^*) be two groups, then the mapping φ: (G, f) → (G^*, f^*) is termed as homomorphism if

$$\varphi(f(a,b) = f^*[\varphi(a), \varphi(b)] .$$

Denoting f and f^* by $\cdot\, and\, *$ respectively we have $\varphi(a\cdot b) = [\varphi(a) * \varphi(b)]$ where f(a, b) = $(a\cdot b)$ $f^*[\varphi(a), \varphi(b)] = \varphi(a) * \varphi(b)$ i.e., $f^*[\varphi(a), \varphi(b)] = a^* * b^*$

Definition 1.5: Stabilizer of an element under the action of a group

Suppose G acts on a set X and we have x ∈ X, then we define Stab x = {g ∈ G | g(x) = x} which is found to be a subgroup of G, called the *Stabilizer of x*. Thus these are the set of elements of G that fix x.

Example1.5: Let X = $\{x_1, x_2\}$ be any set and G = $\{g_1, g_2(= e)\}$ be a group such that G acts on X.

If $g_1(x_1) = x_2$ $g_2(x_1) = x_1$ $g_2(x_2) = x_2$

Thus Stab $(x_1) = \{g_2\}$ and Stab $(x_2) = \{g_2\}$

Definition 1.6: The Orbit of an element under action of a group:

This includes the set of all images of an element x ∈ X under the action of a group G, referred to as the orbit of x under G and denoted by Orb x = {g(x) | g∈G}.

Example 1.6: Let X = $\{x_1, x_2\}$ be any set and G = $\{ g_1, g_2(= e)\}$ be a group such that G acts on X.
Let $g_1(x_1) = x_2$ $\quad g_2(x_1) = x_1$ $\quad g_2(x_2) = x_2$
Then Orb (x_1) = $\{g_1(x_1), g_2(x_1)$ / g∈G} and Orb (x_2) = $\{g_2(x_2)$ / g∈G}

Definition 1.7: Orbit-Stabilizer theorem
This theorem states that - If G is a group which acts on a finite set X with Orb(x) and Stab(x) denoting the orbit and stabilizer of an element 'x' respectively, then

$$|Orb(x)|\,|Stab(x)| = |G|$$

SECTION 2:
In this section we generalize the definition of action of a group to general sets.

Mapping as Action
Any mapping $f : A \rightarrow B$ can be interpreted as an action of 'f' on A. Instead of looking for mappings elsewhere, we can interpret the elements of A as mappings from $B \rightarrow B$. This may be called one-point basic action on B which is now being defined formally.

Definition 2.1: One-point Basic action(OBA)
A non empty set X is said to be a one point basic action on a non empty set Y if for each x ∈ X, x:Y → Y is a mapping
Example 2.1: Let X = $\{ x_1, x_2, x_3 \}$ and Y = $\{ y_1, y_2, y_3\}$
$x_1(y_1) = y_2$; $x_2(y_1) = y_2$; $x_3(y_1) = y_2$;
$x_1(y_2) = y_1$; $x_2(y_2) = y_3$; $x_3(y_2) = y_3$;
$x_1(y_3) = y_3$; $x_2(y_3) = y_1$; $x_3(y_3) = y_1$;
are few possible one-point basic actions on set Y.

Definition 2.2: Two-point Basic action(TBA)
A two point basic action of X on Y is a mapping x : Y × Y → Y for each x∈X i.e., every element of X is a projection.
Example 2.2: Let X = $\{ x_1, x_2, x_3 \}$ and Y = $\{ y_1, y_2, y_3\}$
$x_1(y_1, y_2) = y_2$; $x_2(y_1, y_2) = y_2$; $x_3(y_1, y_2) = y_2$;
$x_1(y_2, y_3) = y_1$; $x_2(y_2, y_3) = y_3$; $x_3(y_2, y_3) = y_3$;
$x_1(y_3, y_1) = y_3$; $x_2(y_3, y_1) = y_1$; $x_3(y_3, y_1) = y_1$;
are some possible two-point basic actions on set Y.

Definition 2.3: Three point basic action-(ThBA)
If for each x ∈ X, x: (Y × Y× Y) → Y then X is called a three-point basic action.
Example 2.3: $x_1(y_1, y_2, y_3) = y_2$;

We can extend this to any finite cross products of Y. Notice here that while one-point basic action is a simple map, a two-point basic action is a projection on Y. The above definitions can be so defined that it leads to group action on a set. Also notice here that the above definitions are as in the definition of a group.

SECTION 3
We earlier defined one-point and two-point basic actions. We now use them to define an action. To define this action, we first use one-point basic action.

Definition 3.1: One-point Action(OPA)
X is called a one-point action on Y if ∃ a mapping $\psi : X \times Y \rightarrow Y$ such that

(i) $\psi(e, y) = y$ for some e ∈ X, ∀ y ∈ Y
(ii) $\psi(x_1(x_2), y) = \psi(x_1, \psi(x_2, y))$, ∀ x_1, x_2 ∈ X, ∀ y ∈ Y

where X is a basic one-point action on X, i.e., x: X → X is an OBA on X.

Example 3.1: Let X = { x_1 , x_2, x_3 } and Y = { y_1, y_2, y_3 }
$\psi(x_1, y_1) = y_2$; $\psi(x_2, y_1) = y_2$; $\psi(x_3, y_1) = y_1$; $\psi(x_1, y_2) = y_3$; $\psi(x_2, y_2) = y_3$; $\psi(x_3, y_2) = y_3$; $\psi(x_1, y_3) = y_1$; $\psi(x_2, y_3) = y_1$; $\psi(x_3, y_3) = y_2$ are few possible one-point actions on set Y.

To define two-point action of a set, we use one-point basic action.

Definition 3.2: Two-point Action (TPA)

X is called a two-point action on Y if ∃ a mapping $\psi : (X \times (Y \times Y)) \to Y$ such that

(i) $\psi(e, y) = y$ for some e ∈ X and ∀ y∈ $Y \times Y$

(ii) $\psi(x_1(x_2), y) = \psi(x_1, \psi(x_2, y))$, ∀ y∈ $Y \times Y$ and we have used one-point action of X.

Example 3.2: Let X = {x_1 , x_2, x_3} and Y = {y_1 , y_2, y_3}

$\psi(x_1(x_1), y_1) = y_3$, $\psi(x_1(x_2), y_1) = y_3$, $\psi(x_1(x_3), y_1) = y_2$,
$\psi(x_1(x_1), y_2) = y_1$, $\psi(x_1(x_2), y_2) = y_1$, $\psi(x_1(x_3), y_2) = y_1$,
$\psi(x_1(x_1), y_3) = y_2$, $\psi(x_1(x_2), y_3) = y_2$, $\psi(x_1(x_3), y_3) = y_3$.

are some possible elements in this category.

This definition can be extended to define an n-point action, for instance a three-point action.

Definition 3.3: Three-point Action (ThPA)

X is a three point action on Y if ∃ a mapping ψ: $X \times (Y \times Y \times Y) \to Y$ such that

(i) $\psi(e, y) = y$ for some e ∈ X.

(ii) $\psi(x_1(x_2(x_3)), y_1) = \psi(x_1, \psi(x_2(x_3), y_1))$
$= \psi(x_1, \psi(x_2, \psi(x_3, y_1)))$ ∀ y∈ $Y \times Y \times Y$

Example 3.3: Let X = { x_1 , x_2, x_3 } and Y = { y_1 , y_2, y_3 }
$\psi(x_1(x_2(x_3)), y_1) = y_3$, $\psi(x_1(x_2(x_3)), y_2) = y_2$, $\psi(x_1(x_2(x_3)), y_3) = y_1$.

Thus we see how natural these definitions are. Now we continue our work under the above definitions.

SECTION 4:

In this section we generalize the definition of stabilizer defined in the introduction section.

Definition 4.1: Stabilizer of an element under one-point basic action(OBA) of a set

The stabilizer of x ∈ X under the one-point basic action of X on Y is a subset of X defined as

Stab (y) = {x ∈ X / x(y) = y} ⊂ X, for Stab(y) is a subgroup of X that fixes the elements.

Example 4.1: Let X = { x_1 , x_2, x_3 } and Y = { y_1, y_2, y_3}
$x_1(y_1) = y_3$, $x_2(y_1) = y_1$, $x_3(y_1) = y_2$,
$x_1(y_2) = y_1$, $x_2(y_2) = y_3$, $x_3(y_2) = y_1$,
$x_1(y_3) = y_2$, $x_2(y_3) = y_2$, $x_3(y_3) = y_3$.
⇒ Stab(y_1) = { x_2} Stab(y_2) = ∅ Stab(y_3) = { x_1}

Definition 4.2: Stabilizer for the entire set Y under one-point basic action(OBA) of a set

Considering the one-point basic action of a set i.e., x:Y → Y , we see that the Stabilizer of the set X is defined as

Stab(Y) = {x ∈ X / x(y) = y, ∀ y ∈ Y}

Example 4.2: Let X = { x_1 , x_2, x_3 } and Y = { y_1 , y_2, y_3 }

Case (i):
$x_1(y_1) = y_1$; $x_2(y_1) = y_2$; $x_3(y_1) = y_3$
$x_1(y_2) = y_2$; $x_2(y_2) = y_3$; $x_3(y_2) = y_2$
$x_1(y_3) = y_3$; $x_2(y_3) = y_1$; $x_3(y_3) = y_1$
⇒ Stab(y_1) = { x_1} ; Stab(y_2) = { x_1}; Stab(y_3) = { x_1}.
⇒ Stab(Y) = { x_1 }.

Case (ii):
$x_1(y_1) = y_2$; $x_2(y_1) = y_1$; $x_3(y_1) = y_3$;
$x_1(y_2) = y_1$; $x_2(y_2) = y_3$; $x_3(y_2) = y_2$;
$x_1(y_3) = y_3$; $x_2(y_3) = y_2$; $x_3(y_3) = y_1$.
⇒Stab(y_1) = { x_2} Stab(y_2) = { x_3} Stab(y_3) = { x_1 }
⇒Stab(Y) = {x_2, x_3, x_1} = X

Case (iii):

$x_1 (y_1) = y_2$ $x_2 (y_1) = y_3$ $x_3 (y_1) = y_2$

$x_1(y_2) = y_3$ $x_2(y_2) = y_1$ $x_3(y_2) = y_3$

$x_1 (y_3) = y_1$ $x_2(y_3) = y_2$ $x_3(y_3) = y_1$

$\Rightarrow$Stab(y_1) = ∅ Stab(y_2) = ∅ Stab(y_3) = ∅

$\Rightarrow$Stab(Y) = ∅

Thus we have defined stabilizers of a single element and for the entire set under OBA. Physically stabilizer leaves a subset unaltered. We can define action of a set on two or more elements after we define stabilizer of two elements.

Remarks:

1. It is also possible that the Stabilizer for the entire set is null. In example 4.1, we see that Stab(Y) = $\{x_1,x_2\}$. Hence the Stabilizer for the entire set may comprise of zero, one, two,…,or n elements depending on the number(n) elements of set X. Thus **max** $|\boldsymbol{Stab}\ (\boldsymbol{Y})| = |\boldsymbol{X}|$ **and min** $|\boldsymbol{Stab}\ (\boldsymbol{Y})| = \emptyset$
2. Also if the elements of Y are mapped to itself, all such mappings or elements of X serve as the ***set identity*** element similar to the group identity element.
3. Only when the stabilizer for each individual element is a null set, the stabilizer of the entire set is null.

Definition 4.3: Stabilizer of an element under one-point action(OA) of a set

Let ψ: X × $Y \rightarrow Y$ be a one-point action on set Y. The stabilizer of an element under this one-point action is defined as

$\Rightarrow$ Stab (y) = {x ∈ X / ψ(x, y) = y }

Example 4.3.1: Let X = { x_1 , x_2, x_3 } and Y = { y_1 , y_2, y_3 }

$\psi (x_1,\ y_1) = y_2$ $\psi (x_2,\ y_1) = y_2$; $\psi (x_3, y_1) = y_1$;

$\psi (x_1,\ y_2) = y_3$ $\psi (x_2,\ y_2) = y_3$ $\psi(x_3,\ y_2) = y_3$;

$\psi (x_1,\ y_3) = y_1$ $\psi (x_2,\ y_3) = y_1$; $\psi(x_3,\ y_3) = y_2$;

$\Rightarrow$ Stab (y_1) = $\{x_3\}$; Stab (y_2) = ∅; Stab (y_3) = ∅

Example 4.3.2: Let us consider the action of sets of integers(Z) on the set of rational numbers(Q). We define this action of Z on Q by the mapping $\varphi : Z^+ \times Q \rightarrow Q$ defined by $\varphi(z, q) = zq$.

(i) In this case the stabilizer of an element, say q_1, is

Stab(q_1) = $\{z \in Z^+ \mid zq_1 = q_1, where\ q_1 \in Q\}$

Stab(q_1) = {1}.

Thus we see that #Stab(q_1) = 1 i.e., the order of the Stabilizer of the element q_1 is equal to one.

(ii) Orbit of an element, say q_1, is

Orb(q_1) = $\{ \varphi(z, q) \mid z \in Z^+\}$

Orb(q_1) = $\{ zq \mid z \in Z^+\}$

Thus we see that # of Orb(q_1) = # of elements in Z^+ i.e., the order of the Orbit of the element q_1 has the same value as that of order of the set Z^+.

Case (i) : If set Z^+ occurs to be of finite order 'n', then the #Orb(q_1) = n.

Case (ii): If the set Z^+ happens to be of infinite order, then the #Orb(q_1) = ∞ (i.e., is infinite)

Example 4.3.3:

We define the action of Z on Q by the mapping $\varphi : Z^\Lambda \times Q \rightarrow Q$ defined as in the below cases.

Here $Z^\Lambda = \{Z^+ + \{0\}\}$

Case 1: If $\varphi(n, q)$ = n(q) = q^n.

In this case,

(1) The stabilizer of an element q is defined as

Stab(q) = { n ∈ $Z^\Lambda \mid \varphi(n, q) = q^n = q\ ,\ q \in Q\}$

This $\Longrightarrow$ Stab(q) = {1}

(2) The orbit of element 'q' is defined as

Orb (q) = { $\psi(z, q)$ | n ∈ Z^Λ}

Orb (q) = $\{q^n$ | n ∈ $Z^\Lambda\} = \{q^0, q^1, q^2, q^3, \ldots\}$

Note: The orbit-stabilizer theorem holds good for this case.

Case 2: If $\varphi(n,q)$ = n(q) = n^q
In this case,

(1) The stabilizer of an element q is described as

Stab(q) = { n ∈ Z^{Λ} | $\varphi(n,q) = n^q = q\ ,\ q \in Q\}$

This ⟹ $n^q = q$

i.e., (q) ∗ log(n) = log(q)

log (n) = $\frac{log(q)}{q}$

⇒ log (n) = $\frac{1}{q}$ ∗ log (q)

⇒ log (n) = log $[(q)^{(\frac{1}{q})}]$

⇒ n = $(\ q\ )^{(\frac{1}{q})}$

∴ Stab(q) = $\{(1/q)^{th}\ roots\ of\ q\}$

(2) The orbit of element 'q' is described as:

Orb (q) = { $\psi(z,q)$ | n ∈ Z^{Λ}}

Orb (q) = $\{n^q$ | n ∈ $Z^{\Lambda}\}$ = $\{q^{th}\ roots\ of\ n, where$ n $\in\ Z^{\Lambda}\ and\ q \in Q\ \}$

Note: The orbit-stabilizer theorem holds good for this case.

Example 4.3.4: We define the action of Z on Q by the mapping $\varphi : Z^{+} \times Q \rightarrow Q$ defined by

$$\psi(z,q) = n^q - q^n\ .$$

1.) Under this action, the stabilizer of the element q is defined as

Stab(q) = { n ∈ Z^{+} | $n^q - q^n = q\ ,\ q \in Q\}$

Here we need to find when this condition $n^q - q^n = q$ is applicable.
Taking log on both sides, we have (q) ∗ log (n) – n log (q) = log (q)
q∗log (n) = log q + n log (q) ⇒ q∗log (n) = log (q) + log (q^n)

⇒ q ∗ log (n) = log [q∗(q^n)]

⇒ q ∗ log (n) = log (q^{n+1})

⇒ log(n) = $\frac{1}{q}$ log(q^{n+1})

⇒ log n = log $(q^{n+1})^{\frac{1}{q}}$

⇒ n = $(q^{n+1})^{\frac{1}{q}}$

2.) The orbit of the element 'q' is stated as

Orb (q) = { $\psi(z,q)$ | n ∈ Z^{+}}

Orb (q) = $\{n^q - q^n$ | n ∈ $Z^{+}\}$

For eg: Orb($\frac{1}{2}$) = {-1, $\frac{1}{2}$, $\sqrt{2} - \frac{1}{4}$, $\sqrt{3} - \frac{1}{8}$, ….}.

We infer that Orb($\frac{p}{q}$) = selected elements of Q as per the definition of $\psi(z,q)$.

Definition 4.4: Stabilizer for the entire set Y under one-point action(OA) of a set
Let ψ: X × $Y \rightarrow Y$ be a one-point action on set Y. The stabilizer of an element under this one-point action is defined as:

Stab (Y) = {x ∈ X / ψ (x, y) = y, ∀ y ∈ Y}

Example 4.4: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$\psi\ (x_1, y_1) = y_1$; $\psi\ (x_2, y_1) = y_2$; $\psi\ (x_3, y_1) = y_1$;
$\psi\ (x_1, y_2) = y_3$; $\psi\ (x_2, y_2) = y_3$; $\psi\ (x_3, y_2) = y_3$;
$\psi\ (x_1, y_3) = y_2$; $\psi\ (x_2, y_3) = y_1$; $\psi\ (x_3, y_3) = y_3$.

⇒Stab(y_1) = {x_1}; Stab(y_2) = ∅; Stab(y_3) = ∅

⇒Stab(Y) = {x_1}

Definition 4.5: Stabilizer of an element under two-point action(TA)

Let ψ: X $\times Y \times Y \rightarrow Y$ be a two-point action on set Y. The stabilizer of an element under this two-point action is defined as

$\Rightarrow$ Stab (y) = $\{x_1(x_2) \in$ X / $(x_1(x_2), y) = y$, $\forall$ y$\in Y \times Y$; $\forall$ x $\in$ X } and we use one point action of X.

Example 4.5: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$\psi (x_1(x_1),\ y_1) = y_3$, $\psi (x_1(x_2),\ y_1) = y_3$, $\psi (x_1(x_3),\ y_1) = y_2$,
$\psi (x_1(x_1),\ y_2) = y_1$, $\psi (x_1(x_2),\ y_2) = y_1$, $\psi (x_1(x_3),\ y_2) = y_1$,
$\psi (x_1(x_1),\ y_3) = y_2$, $\psi (x_1(x_2),\ y_3) = y_2$, $\psi (x_1(x_3),\ y_3) = y_3$.

are some possible elements. The Stabilizer for the above set of elements is defined as

Stab(y_1)= $\emptyset$; Stab(y_2)= $\emptyset$; Stab(y_3)= $\{x_1(x_3)/\ \psi\ (x_1(x_3), y_3) = y_3,\ x_1, x_3 \in$ X; $y_3 \in Y$ }

Definition 4.6: Stabilizer of two points

Let x: Y $*$ Y $\rightarrow$ Y $*$ Y and let ψ: X $\times\ Y\ \times\ Y\ \rightarrow\ Y$ with $(y_1\, y_2\,) \in$ Y $*$ Y, then the stabilizer of the element $(y_1\, y_2\,)$ under basic one-point action is defined as:

$$\text{Stab}\ (y_1 , y_2) = \{\ x \in X\ /\ x(y_1 , y_2) =\ (y_1 , y_2) \in Y * Y\ \}$$

Observe here that for one-point and two-point basic action on Y we consider elements of X as mappings from $Y \rightarrow Y$ or $Y \times Y \rightarrow Y$ etc., where as when we define this using one-point action on Y, we take the mapping ψ: X $\times Y \rightarrow Y$. Hence for a two-point action we take the mapping ψ: X $\times Y \times Y \rightarrow Y$ for which the elements of X are defined as mappings from Y $*$Y $\rightarrow$ Y $*$Y with stabilizer of two points fixing the two points as per the definition. Therefore, under the basic one-point action

Stab $(y_1 , y_2 , y_3 , y_4\ \dots\ y_n)$ = {x $(y_1 , y_2 , y_3 , y_4\ \dots\ y_n)$ = $(y_1 , y_2 , y_3 , y_4\ \dots\ y_n)$ / x $\in$ X}

And under the one-point action we have

Stab $(y_1 , y_2 , y_3 , y_4\ \dots\ y_n)$ = {ψ(x, $(y_1 , y_2 , y_3 , y_4\ \dots\ y_n)$) = $(y_1 , y_2 , y_3 , y_4\ \dots\ y_n)$ / x $\in$ X}

This implies that Stab (for a$\in Y$) is equal to a subset of X such that every element of X maps a to a i.e., x(a) = a. Also a(a) = a. Thus Stabilizer of an element physically means that the action does not disturb that element.

Remark: If x $\cdot$ y = y $\Rightarrow$ x= e. In case of a group, the stabilizer comprises of only 'e'.

SECTION 5

This section deals with orbits in one-point and two-point. Group actions on a set X can be looked into as either

- Group actions of homomorphism or bijective permutations from set X $\rightarrow$ X,
  OR
- Are simply the equivalence relations on X or otherwise the partitions of a set. The equivalence classes we deal with group actions are called orbits.

Definition 5.1: Orbit of an element 'y' under OBA

The orbit of an element is defined as the set of all images in set Y under the one-point basic action of the set X.

$$\text{i.e.,}\quad \text{Orb(y)} = \{x(y) \in Y\ /\ x \in X,\ y \in Y\}$$

Example 5.1: Let X = $\{x_1, x_2, x_3\}$ and Y = $\{y_1, y_2, y_3\}$

$x_1 (y_1) = y_1$, $x_2 (y_1) = y_3$, $x_3 (y_1) = y_3$,
$x_1 (y_2) = y_3$, $x_2 (y_2) = y_1$, $x_3 (y_2) = y_2$,
$x_1 (y_3) = y_2$, $x_2 (y_3) = y_2$, $x_3 (y_3) = y_1$.

Then Orb (y_1) = $\{x_1 (y_1)\ / x_1 \in$ X}; Orb (y_2) = { ($x_3 (y_2)$ / $x_3 \in$ X } ; Orb (y_3) = $\emptyset$

Definition 5.2: Orbit for the entire set Y under one-point basic action (OBA)

The orbit of the entire set Y under one-point basic action is defined as

$$\text{Orb}\ (Y) = \{\ x(y) \in Y\ /\ x \in X,\ \forall y \in Y\ \}$$

Example 5.2: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$x_1(y_1) = y_1$, $x_2(y_1) = y_3$, $x_3(y_1) = y_3$,

$x_1(y_2) = y_3$, $x_2(y_2) = y_1$, $x_3(y_2) = y_2$,

$x_1(y_3) = y_2$, $x_2(y_3) = y_2$, $x_3(y_3) = y_1$.

Thus Orb (y_1) = { $x_1(y_1)$ / $x_1 \in$ X}; Orb (y_2) = { ($x_3(y_2)$ / $x_3 \in$ X}; Orb (y_3) = ∅

∴ Orb(Y) = { $x_1(y_1)$, $x_3(y_2)$ / $x_1, x_3 \in$ X }

Definition 5.3: Orbit of an element 'y' under TBA

We describe the Orbit of an element under the two-point basic action of the set X as

Orb (y_1, y_2) = { x(y_1, y_2) / x ∈ X, (y_1, y_2) ∈ Y ×Y}

Example 5.3: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$x_1(y_1, y_2) = y_2$; $x_2(y_1, y_2) = y_2$; $x_3(y_1, y_2) = y_2$;

$x_1(y_2, y_3) = y_1$; $x_2(y_2, y_3) = y_3$; $x_3(y_2, y_3) = y_3$;

$x_1(y_3, y_1) = y_3$; $x_2(y_3, y_1) = y_1$; $x_3(y_3, y_1) = y_1$.

Thus Orb (y_1) = { $x_2(y_3, y_1)$, $x_3(y_3, y_1)$ / $x_1, x_3 \in$ X };

Orb (y_2) = { $x_1(y_1, y_2)$, $x_2(y_1, y_2)$, $x_3(y_1, y_2)$ / $x_1, x_2, x_3 \in$ X };

Orb (y_3) = { $x_2(y_2, y_3)$, $x_3(y_2, y_3)$ / $x_2, x_3 \in$ X }

Definition 5.4: Orbit for the entire set Y under TBA

The orbit for the entire set under the two-point basic action of the set X is defined as

Orb (Y) = {x(y_*, y_*)/ x ∈ X, (y_*, y_*) ∈ Y ×Y}

Example 5.4: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$x_1(y_1, y_2) = y_2$; $x_2(y_1, y_2) = y_2$; $x_3(y_1, y_2) = y_2$;

$x_1(y_2, y_3) = y_1$; $x_2(y_2, y_3) = y_3$; $x_3(y_2, y_3) = y_3$;

$x_1(y_3, y_1) = y_3$; $x_2(y_3, y_1) = y_1$; $x_3(y_3, y_1) = y_1$;

Thus Orb (y_1) = { $x_2(y_3, y_1)$, $x_3(y_3, y_1)$ / $x_1, x_3 \in$ X };

Orb (y_2) = { $x_1(y_1, y_2)$, $x_2(y_1, y_2)$, $x_3(y_1, y_2)$ / $x_1, x_2, x_3 \in$ X };

Orb (y_3) = { $x_2(y_2, y_3)$, $x_3(y_2, y_3)$ / $x_2, x_3 \in$ X }

Hence Orb (Y) = { $x_2(y_3, y_1)$, $x_3(y_3, y_1)$, $x_1(y_1, y_2)$, $x_2(y_1, y_2)$, $x_3(y_1, y_2)$, $x_2(y_2, y_3)$, $x_3(y_2, y_3)$ }

Definition 5.5: Orbit of an element under one-point action (OPA)

Let ψ: X × $Y \rightarrow Y$ be a one-point action on set Y. The orbit of an element under this one-point action is defined as Orb (y) = { ψ(x, y) / x ∈ X, y ∈ Y }

Example 5.5.1: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$\psi(x_1, y_1) = y_2$; $\psi(x_2, y_1) = y_3$; $\psi(x_3, y_1) = y_1$;

$\psi(x_1, y_2) = y_3$; $\psi(x_2, y_2) = y_2$; $\psi(x_3, y_2) = y_3$;

$\psi(x_1, y_3) = y_1$; $\psi(x_2, y_3) = y_1$; $\psi(x_3, y_3) = y_2$;

Thus Orb (y_1) = { $\psi(x_3, y_1)$ / $x_3 \in$ X}; Orb (y_2) = { $\psi(x_2, y_2)$ / $x_2 \in$ X } ; Orb (y_3) = ∅

Example 5.5.2: Let us consider the action of sets of integers(Z) on the set of rational numbers(Q). We define this action of Z on Q by the mapping $\varphi : Z^+ \times Q \rightarrow Q$ defined by $\varphi(z, q) = zq$.

(i) Orbit of an element, say q_1, is

Orb(q_1) = { $\varphi(z, q)$ | $z \in Z^+$}

Orb(q_1) = { zq | $z \in Z^+$}

Thus we see that # Orb(q_1) = # of elements on Z^+ i.e., the order of the Orbit of the element q_1 is equivalent to the order of the set Z^+.

Case (i) : If set Z^+ is chosen to be of finite order 'n', then the #Orb(q_1) = n.

Case (ii): If the set Z^+ is chosen to be of infinite order, then the #Orb(q_1) = ∞ (i.e., is infinite)

Example 5.5.3: We define the action of Z on Q by the mapping $\varphi : Z^{\Lambda} \times Q \rightarrow Q$ defined as in the below cases. Here $Z^{\Lambda} = \{Z^+ + \{0\}\}$

Case 1: If $\varphi(n, q)$ = n(q) = q^n.

The orbit of an element q is defined as

Orb(q) = { $\psi(z,q)$ | n ∈ Z^{Λ}}

Orb(q) = {q^n | n ∈ Z^{Λ}} = {$q^0, q^1, q^2, q^3, \ldots$}

Note: The orbit-stabilizer theorem holds good for this case.

Case 2: If $\varphi(n,q)$ = n(q) = n^q

In this case,

The orbit of an element q is defined as

Orb(q) = { $\psi(z,q)$ | n ∈ Z^{Λ}}

Orb(q) = {n^q | n ∈ Z^{Λ}} = {q^{th} *roots of n, where* n ∈ Z^{Λ} *and* $q \in Q$ }

Note: The orbit-stabilizer theorem holds good for this case.

Example 5.5.4: We define the action of Z on Q by the mapping $\varphi : Z^+ \times Q \rightarrow Q$ defined by $\psi(z,q) = n^q - q^n$.

The orbit of an element q is defined as

Orb(q) = { $\psi(z,q)$ | n ∈ Z^+}

Orb(q) = { $n^q - q^n$ | n ∈ Z^+}

For eg: Orb($\frac{1}{2}$) = { -1, $\frac{1}{2}$, $\sqrt{2} - \frac{1}{4}$, $\sqrt{3} - \frac{1}{8}$, …. }

Definition 5.6: Orbit for the entire set Y under one-point action (OPA)

Let ψ: X × $Y \rightarrow Y$ be a one-point action on set Y. The orbit of an element under this one-point action is defined as: Orb (Y) = { ∪ { ψ(x, y)} / x ∈ X and ∀ y ∈ Y }

Example 5.6: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$\psi(x_1, y_1) = y_2$; $\psi(x_2, y_1) = y_3$; $\psi(x_3, y_1) = y_1$;
$\psi(x_1, y_2) = y_3$; $\psi(x_2, y_2) = y_2$; $\psi(x_3, y_2) = y_3$;
$\psi(x_1, y_3) = y_1$; $\psi(x_2, y_3) = y_1$; $\psi(x_3, y_3) = y_3$.

Thus Orb (y_1) = { $\psi(x_3, y_1)$ / x_3 ∈ X}; Orb(y_2)={ $\psi(x_2, y_2)$ / x_2 ∈ X } ; Orb (y_3) = ∅

Thus Orb (Y) = { $\psi(x_2, y_2)$, $\psi(x_3, y_1)$, $\psi(x_3, y_3)$ / x_2, x_3 ∈ X; ∀ y ∈ Y }

Definition 5.7: Orbit of an element under two-point action (TPA)

Let x: Y × Y → Y × Y and let ψ: X × $Y \times Y \rightarrow Y$ with (y_1, y_2) ∈ Y × Y, then the orbit of the element (y_1, y_2) under the two-point action is defined as

Orb (y_1 , y_2) = { ψ(x, y, y) / x ∈ X, y = (y_1, y_2) ∈ Y × Y }

Example 5.7: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$\psi(x_1, y_1, y_2) = y_2$; $\psi(x_2, y_1, y_2) = y_3$; $\psi(x_3, y_1, y_2) = y_1$;
$\psi(x_1, y_1, y_3) = y_3$; $\psi(x_2, y_1, y_3) = y_2$; $\psi(x_3, y_1, y_3) = y_2$;
$\psi(x_1, y_2, y_3) = y_1$; $\psi(x_2, y_2, y_3) = y_1$; $\psi(x_3, y_2, y_3) = y_3$.

Thus Orb (y_2) = { $\psi(x_1, y_1, y_2)$ / x_1 ∈ X}; Orb (y_2) = ∅ ; Orb (y_3) = { $\psi(x_3, y_2, y_3)$ / x_3 ∈ X }

Definition 5.8: Orbit of the entire set Y under two-point action (TPA)

The definition of orbit under the action of a set does not change under the basic action.

Orb (Y) = { ∪{ ψ(x, y) } / x ∈ X, ∀ (y_1, y_2) ∈ Y × Y }

Example 5.8: Let X = { x_1, x_2, x_3 } and Y = { y_1, y_2, y_3 }

$\psi(x_1, y_1, y_2) = y_2$; $\psi(x_2, y_1, y_2) = y_3$; $\psi(x_3, y_1, y_2) = y_1$;
$\psi(x_1, y_1, y_3) = y_3$; $\psi(x_2, y_1, y_3) = y_2$; $\psi(x_3, y_1, y_3) = y_2$;
$\psi(x_1, y_2, y_3) = y_1$; $\psi(x_2, y_2, y_3) = y_1$; $\psi(x_3, y_2, y_3) = y_3$.

Thus Orb (y_2) = { $\psi(x_1, y_1, y_2)$ / x_1 ∈ X}; Orb (y_2) = ∅; Orb (y_3) = { $\psi(x_3, y_2, y_3)$ / x_3 ∈ X }

Orb(Y) = { $\psi(x_1, y_1, y_2)$, $\psi(x_3, y_2, y_3)$ / x_1, x_3 ∈ X }

Definition 5.9: Orbit of two points

Orb (y_1, y_2) = {($y_1 y_2$) / x ∈ X, x($y_1 y_2$) ∈ Y ∗ Y }

= { ($x(y_1)$, $x(y_2)$) / ∀ x ∈ X }

Also, Orb ($y_1, y_2, y_3, y_4 \ldots y_n$) = {∪(x($y_i$)) / x ∈ X , i=1,2,3,4…n}

If $y_1, y_2, y_3, y_4 \ldots y_n$ ∈ Y then we interpret elements of x as mappings from

$$y_1 \times y_2 \times y_3 \times y_4 \times \ldots \times y_n \rightarrow y_1 \times y_2 \times y_3 \times y_4 \times \ldots \times y_n$$

By definition of an orbit under TBA, every element of X serves to be a mapping from $Y * Y \rightarrow Y$. Therefore for each element in X there is a corresponding set of mappings in Y. In general, these sets do not intersect as the results of the below two cases:

- Case 1: If there are three elements in the first set X and an equal number of elements appear in the second set Y, subsequently the orbits may be defined as below:

$x_1(y_1, y_2) = y_1$; $x_2(y_1, y_2) = y_1$; $x_3(y_1, y_2) = y_1$;
$x_1(y_2, y_3) = y_2$; $x_2(y_2, y_3) = y_2$; $x_3(y_2, y_3) = y_2$;
$x_1(y_3, y_1) = y_3$; $x_2(y_3, y_1) = y_3$; $x_3(y_3, y_1) = y_3$.

- Case 2: If the number of members in set X is either equal to or less than the number of members in set Y also we get the result as in case 1.
- Case 3: Suppose we now take a subset of Y represented as Y', say having 4 elements. The set Y contains 7 elements. It follows that

$Y \times Y = \{ (y_1, y_2)(y_2, y_3)(y_3, y_4)(y_4, y_5)(y_5, y_6)(y_6, y_7)(y_7, y_1) \}$
$Y' = \{ (y_2, y_3)(y_3, y_4)(y_4, y_5)(y_5, y_6)(y_6, y_7)(y_7, y_1) \}$ where
$x_1(y_i, y_j) = \{$any element other than y_i and y_j, j=i+1 $\}$

Hence the mappings under the functions x's result in disjoint sets. The unification of these disjoint sets may be equal to the set Y. In order that the union of these disparate sets is equal to set Y, the cardinality of elements in Y' has to be atleast equal to the cardinality of elements in set Y-the union of which will be set Y.

Result: We observe that the union of disparate sets becoming equal to the set Y depends on two conditions:

1) The cardinality of elements in set Y
2) The manner in which we define the set Y.

SECTION 6:

It includes recalling certain classical theorems on orbits and stabilizers, their interpretation for action under basic one-point and two point actions. In mathematics, one notices that two different objects share some internal structure. A homomorphism is a way of translating one object into another that reflects its underlying structure. It is like a mathematical analogy. A homomorphism from X to Y means "Y is an analogy for X in a highly specific way" i.e. all the elements of X have a direct translation into Y. Here the exactness of the structure is not relevant. Though these examples are not serious enough, however, just to feel the new definitions we give the following elementary examples. Which are applicable to practical situations.

Definition 6.1: Set Homomorphism

Let X and Y be two non-empty sets. A mapping φ: $X \rightarrow Y$ is called a 'set homomorphism' if $\varphi(g(a,b)) = g^*(\varphi(a), \varphi(b))$ where g: $X \times X \rightarrow X$ and $g^*: (Y \times Y) \rightarrow Y$, $\forall g \in X$, $\forall g^* \in Y$ i.e. elements of X and Y are respectively two-point basic actions on X and Y.

Proposition 6.1:

Let (G, f) be the group acting on the set X. Then there is a homomorphism χ : G → S(X) defined by $\chi(g(x)) = g(x)$ where g is a one-point basic action on X and 'g' is a bijection on X.

Proof: Let $(g) = g : X \rightarrow X$ be a bijection

Then $\chi(f(g_1, g_2)) = \chi((g_1 g_2), x) : X \times X \rightarrow X$ is also a bijection defined by

$$\chi(f(g_1, g_2))(x) = (g_1 g_2))(x)$$
$$= (g_1, \chi(g_2(x)))$$

Let $\chi(g_2(x))$ = y, then we have

$$\chi(f(g_1, g_2)) = \chi(g_1, y)$$
$$= \chi(g_1)(y)$$
$$= \chi(g_1) \circ (\chi(g_2(x))$$
$$= [\chi(g_1) \circ (\chi(g_2)](x)$$

Remark: Thus in the usual group action, we have a morphism χ : G → S(X) as described above. If G is a group then we have mappings f : G × G → G. If G is not a group we have a two point action from g: X × X → X. We use this to define the set homomorphism i.e. the mapping χ: G → S(Y) is a morphism Treating S(Y) as a set Z we have mapping from G → Z, where G is some set here.

If X and Y are two sets where X acts on set Y, then the mapping χ: X → S(Y) is a morphism. To define this mapping χ: X → S(Y) we note that $\chi(x)$: Y → Y, is a bijection. If we let $\chi(x) = x$, where x is a one-point basic action on Y and also a bijection i.e., $\chi(x) = x$: Y → Y

Hence $\chi(x_1(x_2)) = \chi(x_1) \circ \chi(x_2)$. This is a homomorphism by definition always.

Proof: $\chi(g(x_1, x_2)) = \chi((x_1 x_2), x)$: X × X → X is a bijection defined as

$\chi(g(x_1, x_2))$ (x) = $\chi(g_1, \chi(g_2))$(x)

Consider $\chi(x(x_1), x(x_2))$ where x:Y×Y → Y which is a two-point action of X on Y.

Then $\chi(x(x_1), x(x_2)) = \chi\big(x (x_1(x_2))\big) = x(\chi\big(x_1(x_2)\big)) = x(\chi(x_1), \chi(x_2))$

A common way of specifying non-canonical actions is to describe a morphism from a set X to the group of symmetries of set Y.

Proposition 6.2:

If X acts on the set Y, then the function χ: X → S(Y) is a morphism such that

$$\chi[\varphi \{(g_1(a,b), g_2(a,b))\}]) = \chi[g_1{}^*\big(\varphi(a),\varphi(b)\big), g_2{}^*\big(\varphi(a),\varphi(b)\big)]$$
$$= \chi[g_1{}^*\big(\varphi(a), \varphi(b)\big)] \ \chi[g_2{}^*\big(\varphi(a), \varphi(b)\big)]$$

Section 7: Stability of Action

Structural stability of an element is the application of small perturbations that do not change the element's behavior. This is the study dealt in Control theory. A structurally stable element is an element having a neighborhood contained in its equivalence class that is to say a small perturbation of it gives rise to an element equivalent to it.

In our case of actions on sets, the perturbation is also an element of the set giving rise to an element equivalent to it.

Let ψ: G × X → X be an action. Let S ⊆ X, then ψ is stable on S if $\psi(g \times s) \subseteq$ S.

Let ψ: X × Y → Y exist as a one-point action(OA) if S ⊆ Y and such that if $\psi(x \times s) \subseteq$ S, then ψ is stable on S

Conclusion:

The application of group action is naturally relevant in Physics and Chemistry. Our motivation to study similar application to areas not having classical mathematic structures was essentially the reason to study general action of sets.

With the idea of finding applications of action of sets to such areas like Economics, Finance and Graphs we have generalized some classical concepts and theorems to general sets. The application of these will be taken up elsewhere.

Reference: Book titled "Modern Algebra with Applications" authored by William J Gilbert.