Some Conclusions of Contractions on Complete Metric Space

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Abstract : Fixed point theorems relates to maps f of a set X into itself which, under certain conditions, admit a fixed point, that is, a point x ∈ X such that f(x) = x. Much research has been carried out to find conditions on f and X, sufficient to ensure the existence of a fixed point of f in X and in uniqueness and in procedures for the calculation of fixed points. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology.

In this paper, a theorem of the approach of contractions on complete metric space is presented, which extends fixed point theorems given by Meir and Keeler [5], Edelstein [2], Jungck [3], Park [6] and Chung [1] and few conclusions of the result are reviewed.

***IndexTerms* – Fixed point, contractions, complete metric space.**

I. Introduction

Suppose (X, d) is a metric space. A mapping g: X → X is said to be a self map of X and it is called a contraction if there is a positive real number $\alpha \le 1$ such that $d(gx, gy) \le \alpha\, d(x, y)$ for all x, y ϵ X.

The well known Banach contraction principle states that " if (X,d) is a complete metric space and g is a contraction on X, then g has a unique fixed point ξ". This principle has been generalized in many ways and by several authors.

Park and Bae [6] have generalized the notion of (ε, δ) – contraction using a continuous self map f of a metric space (X, d) as follows:

Suppose (X. d) is a metric space and f is a continuous self mapping of X, then a self map g of X is called as (ε, δ)-f contraction if for any ε > 0 there is a δ > 0 such that for all x,y ϵ X

(1.1) $\varepsilon \le d(fx, fy) < \varepsilon + \delta$ implies $d(gx, gy) < \varepsilon$
and gx = gy whenever fx = fy

Observe that (ε, δ)- contraction of a metric space X is (ε, δ)-I contraction where I is the identity map of X.

Let C_f denote the class of self maps g of X such that fg = gf and g(X) ⊆ f(X), where f is a self map of a metric space (X, d). i.e., $C_f = \{ g : X \to X \mid fg = gf \text{ and } g(X) \subseteq f(X) \}$

Suppose g ϵ C_f. For any x_0 ϵ X, we have $g(x_0)$ ϵ g(X) ⊆ f(X) so that there is x_1 ϵ X with $f(x_1) = g(x_2)$. In this way to each x_0 ϵ X we can find a sequence $\{x_n\}_{n=1}^{\infty}$ such that $fx_{n+1} = gx_n$ for n ≥ 0. Then

(1.2) the sequence $\{fx_n\}_{n=1}^{\infty}$ is called the f iteration of x_0 under g.

In this paper, a common fixed point theorem of park and Bae [6] is discussed for (ε, δ)-f contraction g of X which commutes with f on the complete metric space X and subsequently various conclusions of this result are surveyed.

II. PRELIMINARIES

2.1 Lemma: Let f be a continuous self map of a metric space X and g be an (ε, δ)-f contraction. If x_0 ϵ X and $\{fx_n\}_{n=1}^{\infty}$ is an f-iteration of x_0 under g, then $\{ d(fx_n, fx_{n+1}) \mid n \ge 0 \}$ is monotone decreasing sequence and converges to 0.

Proof: Suppose x_0 ϵ X and $\{ fx_n \}$ is an f-iterate of x_0 under g.

Here g is an (ε, δ) - f contraction. Therefore $d(fx_{n+1}, fx_{n+2}) = d(gx_n, gx_{n+1}) < d(fx_n, fx_{n+1})$, proving that $\{d(fx_n, fx_{n+1})\}_{n=1}^{\infty}$ is a decreasing sequence.

(2.2) Let $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = r,$

If r > 0, then by (1.1), there is a δ > 0 such that $r \le d(fx, fy) < r + \delta$ which implies $d(gx, gy) < r$ for all x,y ϵ X.

Now from (2.2), we get a natural number N such that for every m ≥ N. we have

(2.3) $r \le d(fx_n, fx_{n+1}) < r + \delta,$

then for every m ≥ N we have $d(fx_{m+1}, fx_{m+2}) = d(gx_m, gx_{m+1}) < r$, which contradicts (2.3), proving r = 0.

2.4 Definition: Let (X, d) be a metric space and f, g be two self mappings of X then a point ξ ϵ X is called a coincidence point of f and g if f ξ = g ξ.

2.5 Lemma: Let (X, d) be a metric space, f is a continuous self map of X and g be an (ε, δ)-f contraction. If f and g are two commuting maps having a coincident point at ξ ϵ X, then f ξ (= g ξ) is the unique common fixed point of f and g.

Proof: Given that ξ ϵ X is the coincidence point of f and g so that f ξ = g ξ. Suppose f ξ = η. We now prove f η = η and that η is unique.

If possible f η ≠ η, using g is an (ε, δ)-f contraction, we have d (η, f η) = d (gξ, fgξ) < d (fξ, ffξ) = d (η,f η), which is absurd.

Therefore f η = η.

Similarly, we can show g η = η, proving η is common fixed point of f, g.

Now Let η^1 be a common fixed point of f and g such that $\eta \ne \eta^1$. Since g is an (ε, δ)-f contraction,

$d(\eta, \eta^1) = d(g\eta, g\eta^1) < d(f\eta, f\eta^1) = d(\eta, \eta^1)$, absurd again. Proving η is unique.

2.6 Theorem: ([6], theorem 2.3): Let f be a self map of a metric space (X, d) and g be an (ε, δ)-f contraction commuting with f and g(X) ⊆ f(X). If a point x_0 ϵ X has an f-iteration $\{fx_n\}_{n=1}^{\infty}$ under g with a cluster point ξ ϵ X at which f is continuous, then $\{fx_n\}_{n=1}^{\infty}$ converges to ξ , and f ξ is the unique common fixed point of f and g.

Proof: In view of lemma 2.5 it is sufficient to show that we can find a coincidence point point ξ ϵ X.

(i.e., ξ ϵ X such that f ξ = g ξ).

If d (f x_n, f x_{n+1}) = 0 for some n,

then f x_{n+1} = f x_n, so that g x_n = f x_n giving that x_n is a coincidence point of f and g, proving the theorem with ξ = x_n.

Now suppose d (f x_n, f x_{n+1}) ≠ 0 for any n.

We now claim that $\{fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence. If possible assume that $\{fx_n\}_{n=1}^{\infty}$ is not a Cauchy sequence. Then there exists an ε > 0 and a subsequence $\{fx_{n_i}\}$ of $\{fx_n\}$ such that

(2.7) $d\{fx_{n_i}, fx_{n_{i+1}}\} > 2\varepsilon$.

Now by (1.1), there exists a δ with 0 < δ < ε, such that ε ≤ d (fx, fy) < ε + δ implies d (gx, gy) < ε.

But by lemma 2.1, there exists a positive integer N such that for every m ≥ N.

(2.8) $d\{fx_m, fx_{m+1}\} < \frac{\delta}{6}$

Thus by (2.7) and (2.8) we have for every n_i ≥ N we can find m such that $n_i < m < n_{i+1}$ and

(2.9) $\varepsilon + \frac{\delta}{3} \le d\,(fx_{n_i}, fx_m) < \varepsilon + \delta$

Then $d\,(fx_{n_i}, fx_m) \le \mathrm{d}\,(fx_{n_i}, fx_{n_{i+1}}) + \mathrm{d}\,(fx_{n_{i+1}}, fx_{m+1}) + d\,(fx_{m+1}, fx_m)$

$< \frac{\delta}{6} + d\,(gx_{ni}, gx_m) + \frac{\delta}{6} < \varepsilon + \frac{\delta}{3}$

Which contradicts (2.9). Therefore our assumption is wrong. Hence $\{fx_n\}$ is a Cauchy sequence. Now since $\{fx_n\}_{n=1}^{\infty}$ clusters at ξ ϵ X , it converges to ξ. And as f is continuous at ξ, the sequence { ffx_n } = { fgx_{n-1} } = { gfx_{n-1}} converges to f ξ.

Now if ffx_m = ffx_{m+1} = ffx_{m+2} = … for some m, then { ffx_n } converges to ffx_n and in that case, we have

ffx_m = ffx_{m+1} = fgx_m = gfx_m showing that fx_m = ξ is the coincidence point of f and g and ffx_m = fξ is the coincidence point of f and g and ffx_m = fξ is a common fixed point for f and g by lemma 2.5.

In the case, where there is no integer m satisfying that ffx_m = ffx_{m+1} =… then for any ε > 0, there exists a natural number N such that

(2.10) d (ffx_m = fξ) < ε/2 for m ≥ N.

Now we find an n ≥ N such that

(2.11) ffx_n ≠ fξ.

Then by using (2.10) and the fact that g is an (ε, δ)-f contraction, we get

d (fξ, gξ) ≤ d (fξ, fgx_n) + d (fgx_n, gξ)

= d (fξ, ffx_{n+1}) + d (gfx_n, gξ)

< ε/2 + d (ffx_n, fξ)

< ε/2 + ε/2 = ε

Since ε > 0 is arbitrary, it follows that fξ = gξ.

III. Main Conclusions

3.1 Theorem: let f be a continuous self map of a complete metric space X and g be an (ε, δ)-f contraction in C_f . Then f and g have a unique common fixed point ξ in X. In fact for any x_0 ϵ X every f-iteration of x_0 under g converges to some ξ ϵ X and f ξ = g ξ .

Proof: Since g ϵ C_f, we have by (1.2), that an f-iteration { fx_n } of x_0 under g exists for each x_0 ϵ X. Then by lemma 2.1, { fx_n } is a Cauchy sequence and since X is complete, { fx_n } converges to some ξ ϵ X and f ξ = g ξ.

3.2 Corollary: Let f be a continuous self map of a complete metric space X and g be in C_f. If g^N is an (ε, δ)-f contraction for some positive integer N, then f and g have a unique common fixed point.

Proof: Since g ϵ C_f, clearly it follows that $g^N f = fg^N$ and g^N (X) ⊂ g(X) ⊂ f(X). Therefore g^N ϵ C_f. Hence by Theorem 2.6 we have that g^N and f have a unique common fixed point. Say η. Thus fg η = gf η = g η and g^Ng η = gg^N η = g η showing g η is also a common fixed point of f and g^N which implies g η = η because of uniqueness.

Now suppose η and η^1 are common fixed points of f and g. Then g^N η = η = f η and $g^N \eta^1 = \eta^1$ = f η^1 . Since f and g^N have a common fixed point, we have η = η^1.

3.3 Corollary: If f is a bijective continuous self map of a complete metric space X and for any ε > 0 there exist a δ > 0 such that for all x, y ϵ X, ε ≤ d (fx, fy) < ε + δ implies d (x, y) < ε. Then f has a unique fixed point.

Proof: If we replace g by I_x in Theorem 3.1, the corollary follows.

3.4 Corollary: Let f be a continuous self map of a complete metric space X and $\{ g_\lambda \}_{\lambda\epsilon\Lambda}$ be a family of self maps in C_f . If each g_λ is an (ε, δ)-f contraction, then there exists a unique point η ϵ X such that f η = $g_\lambda\eta$ = η for every $\lambda\epsilon\Lambda$.

Proof: For each $\lambda\epsilon\Lambda$, g_λ and f have a unique common fixed point say η. Then for any $\mu\ \epsilon\ \Lambda$ we have $g_\lambda(g_\mu\eta) = g_\mu(g_\lambda\eta) = g_\mu\eta = g_\mu(f\,\eta) = f\,(g_\mu\eta)$, showing that $g_\mu\eta$ is a common fixed point of g_λ and f. Therefore it follows that $g_\mu\eta$ = η, by uniqueness.

3.5 Corollary: Let f be a self map of a complete metric space X such that f^k is continuous for some positive k. Let g: $f^{k-1}X \to X$ be a map such that $gf^{k-1}X \sqsubseteq f^kX$ and gf = fg, whenever both sides are defined. If gf^{k-1} is an (ε, δ)-f^k contraction then f and g have a unique common fixed point.

Proof: By Theorem 3.1 gf^{k-1}, f^k have a unique common fixed point η.

Then since $gf^{k-1}(f\eta) = g(f^k\eta) = g\eta$ and

$gf^{k-1}(f\eta) = gf(f^{k-1}\eta) = fg(f^{k-1}\eta) = f\eta$

We get $g\eta = f\eta$.

Also we have $f^k(f\eta) = f(f^k\eta) = f\eta$.

Thus $f\eta$ is also a common fixed point of gf^{k-1} and f^k.

Therefore we have $\eta = f\eta = g\eta$, showing η is a common fixed point of f and g. The uniqueness is obvious.

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