Comparative Study of the Homotopy Perturbation Method and Homotopy Perturbation Kamal Transform Method

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Abstract

In this paper, comparative study of homotopy perturbation method (HPM) and homotopy perturbation Kamal transform method built exhibit that Homotopy Perturbation Kamal Transform (HPKTM) method is very rapid convergent to output of the partial differential equation. The solution example demonstrates that the proposed scheme is effective.

Keywords

Homotopy Perturbation Method (HPM), Kamal Transform

1. Introduction

Many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models and chemical kinetics, can be modeled by nonlinear differential equations. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions. Also a new integral transform and some of its fundamental properties are used to solve general nonlinear non-homogenous partial differential equation with initial conditions.The application of the homotopy perturbation method (HPM) in mathematical problems is highly considered by scientists, because without demanding a small parameter in equations, HPM continuously transforms a complex problem which is easy to solve into a simple problem. The homotopy perturbation method [2] was first proposed by He in 1998. It is in fact a coupling of the traditional perturbation method and homotopy in topology. The HPM was further developed and improved by He [2-10] and applied to asymptotology [7], bifurcation for non-linear problems[8], strongly non-linear equations [9] and many other subjects. The method yields a very rapid convergence of the solution series in most cases, and usually only a few iterations lead to very accurate solutions. Although the goal of He's homotopy perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems.

The combinations of homotopy perturbation method and Kamal transform which is studies in this paper. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear partial differential equations.

Kamal transform[22] is derived from the classical Fourier integral. Based on the mathematical simplicity of the Kamal Transform and its fundamental properties, Kamal Transform was introduced by Abdelikh Kamal in 2016, to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform.

2. Homotopy Perturbation Method

To illustrate the basic ideas of the (HPM), we consider the following nonlinear differential equation

$$\mathcal{L}(\omega) + \varkappa(\omega) - f(\beth) = 0 \tag{1}$$

With the boundary condition

$$\beta\left(\omega, \frac{\partial\omega}{\partial n}\right) = 0, \quad \beth \in \lambda, \tag{2}$$

Where $\mathcal{L}$ is linear, while $\varkappa$ is nonlinear, $f(\beth)$ is a know analytic function, λ is the boundary of the domain ϖ .

Define a homotopy $v^{\star}(\beth, \mathcal{P}) : \varpi \times [0,1] \to R$ which satisfics

$$\mathcal{H}(v^{\star}, \mathcal{P}) = (1-\mathcal{P})[\mathcal{L}(v^{\star}) - \mathcal{L}(\omega_0)] + \mathcal{P}[A(v^{\star}) - f(\beth)] = 0, \;\; \mathcal{P} \in [0,1], \; \beth \in \varpi \tag{3}$$

Where $\mathcal{P} \in [0,1]$ is an embedding parameter, ω_0 is an initial approximation of equation (3) then

$$\mathcal{H}(v^{\star}, 0) = \mathcal{L}(v^{\star}) - \mathcal{L}(\omega_0) = 0, \tag{4}$$

$$\mathcal{H}(v^{\star}, 1) = A(v^{\star}) - f(\beth) = 0 \tag{5}$$

The changing process of $\mathcal{P}$ from zero to unity is just that of $v^{\star}(\beth, \mathcal{P})$ from trivial solution $\omega_0(\beth)$ To original solution $\omega_0(\beth)$, in topology this is called deformation, $\mathcal{L}(v^{\star}) - \mathcal{L}(\omega_0)$ and $A(v^{\star}) - f(\beth)$ are called homotopic.

Here the imbedding parameter $\mathcal{P}$ can be considered as "small parameter". Assume that the solution of Equation (3) can be written as a power series in $\mathcal{P}$

$$v^{\star} = v_0^{\star} + \mathcal{P}\, v_1^{\star} + \mathcal{P}^2\, v_2^{\star} + \cdots \tag{6}$$

Setting $\mathcal{P} = 1$ result in the approximate solution of equation (6)

$$\omega = \lim_{\mathcal{P}\to 1} v^{\star} = \; v_0^{\star} + v_1^{\star} + v_2^{\star} + \cdots \tag{7}$$

2. Kamal transform : The Kamal transform is denoted by operator $\mathfrak{K}$ (.) and Kamal transform of $\mathfrak{I}(\tau^*)$ is defined by the integral equation:

$$\mathfrak{K}\big(\mathfrak{I}(\tau^*)\big) = \mathbb{E}(\ell) = \int_0^{\infty} \mathfrak{I}(\tau^*)\, e^{\frac{-\tau^*}{\ell}} d\tau^*, \tau^* \geq 0, \quad \text{and} \quad \Lambda_1 \leq \ell \leq \Lambda_2 . \tag{8}$$

in a set A the function is defined in the form

$$A = \left\{ \mathfrak{I}(\tau^*) : \exists \mathbb{M}, \;\; \Lambda_1, \Lambda_2 > 0 \,.\, |\mathfrak{I}(\tau^*)| < \mathbb{M} e^{\frac{|\tau^*|}{\Lambda_j}}, \text{if } \tau^* \epsilon (-1)^j \times [0, \infty) \right\} \;, \tag{9}$$

where Λ_1 and Λ_2 may be finite or infinite and the constant $\mathbb{M}$ must be finite number.For existence of Kamal transform $\mathfrak{I}(\tau^*)$ is essential for $\tau^* \geq 0$ a piece wise continuous and of exponential order is required , else it does not exist.

Remark : The peruses can be perused more about the Kamal transform in[15-18].

3. Kamal Homotopy Perturbation Strategy (KHPS)

To illustrate the basic idea of this strategy, we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$\mathcal{D}^{\bowtie}\,\omega(\mathfrak{x},\tau^*) + \mathrm{T}\omega(\mathfrak{x},\tau^*) + \varkappa\omega(\mathfrak{x},\tau^*) = \mathcal{g}\,(\mathfrak{x},\tau^*)\,, \tag{10}$$

$$\omega(\mathfrak{x},0) = \hbar(\mathfrak{x})\,,\omega_{\tau^*}(\mathfrak{x},0) = \mathrm{f}(\mathfrak{x})$$

where $\mathcal{D}^{\bowtie}$ is the second order linear differential operator$\mathcal{D}^{\bowtie} = \partial^2/\partial\tau^{*\,2}$, T is the linear differential operator of less order than $\mathcal{D}^{\bowtie}$, $\varkappa$ represents the general non-linear differential operator and $\mathcal{g}\,(\mathfrak{x},\tau^*)$is the source term. Taking the Kamal transform (denoted throughout this paper by $\mathfrak{K}$) on both sides of Eq.(10):

$$\mathfrak{K}[\mathcal{D}^{\bowtie}\,\omega(\mathfrak{x},\tau^*)] + \mathfrak{K}[\,\mathrm{T}\omega(\mathfrak{x},\tau^*)] + \mathfrak{K}[\varkappa\omega(\mathfrak{x},\tau^*)] = \mathfrak{K}[\mathcal{g}\,(\mathfrak{x},\tau^*)]\,. \tag{11}$$

Using the differentiation property of the Kamal transform, we have

$$\frac{1}{\ell^2}\mathfrak{K}[\omega(\mathfrak{x},\tau^*)] - \frac{1}{\ell}\omega(\mathfrak{x},\tau^*) - \omega_{\tau^*}(\mathfrak{x},\tau^*) + \mathfrak{K}[\mathrm{T}\omega(\mathfrak{x},\tau^*)] + \mathfrak{K}[\mathfrak{x}\omega(\mathfrak{x},\tau^*)] = \mathfrak{K}[\mathcal{g}(\mathfrak{x},\tau^*)]$$

$$\mathfrak{K}[\omega(\mathfrak{x},\tau^*)] = \ell\hbar(\mathfrak{x}) + \ell^2\mathrm{f}(\mathfrak{x}) + \ell^2\mathfrak{K}[\mathcal{g}\,(\mathfrak{x},\tau^*)] - \ell^2\mathfrak{K}[\mathrm{T}\omega(\mathfrak{x},\tau^*)] - \ell^2\mathfrak{K}[\varkappa\omega(\mathfrak{x},\tau^*)]\,. \tag{12}$$

Operating with the Kamal inverse transformation on both sides of equation (12) gives

$$\omega(\mathfrak{x},\tau^*) = \mathbb{E}(\mathfrak{x},\tau^*) - \mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}\,[\varkappa\omega(\mathfrak{x},\tau^*) + \mathrm{T}\omega(\mathfrak{x},\tau^*)]\right]. \tag{13}$$

where $\mathbb{E}(\mathfrak{x},\tau^*)$represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation strategy

$$\omega(\mathfrak{x},\tau^*) = \sum_{n=0}^{\infty}\mathcal{P}^n\omega_n\,(\mathfrak{x},\tau^*) \tag{14}$$

and the nonlinear term can be decomposed as

$$\varkappa\omega(\mathfrak{x},\tau^*) = \sum_{n=0}^{\infty}\mathcal{P}^n\,\mathcal{H}_n'(\omega). \tag{15}$$

For some He's polynomials $\mathcal{H}_n'$ (see [19-20]) that are given by

$$\mathcal{H}_n'(\omega_0,\omega_1,\dots,\omega_n) = \frac{1}{n!}\frac{\partial^2}{\partial\mathcal{P}^n}\left[\varkappa\left(\sum_{n=0}^{\infty}(\mathcal{P}^i\omega_i)\right)\right]_{\mathcal{P}=0}\,, n = 0,1,2,3\,\dots$$

Substituting Eq.(15) and (14) in Eq.(13) we get

$$\sum_{n=0}^{\infty}\mathcal{P}^n\,\omega_n(\mathfrak{x},\tau^*) = \mathbb{E}(\mathfrak{x},\tau^*) - \mathcal{P}\left(\mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}\left[\sum_{n=0}^{\infty}\mathcal{P}^n\,\mathcal{H}_n'(\omega) + \mathrm{T}\sum_{n=0}^{\infty}(\mathcal{P}^n\omega_n(\mathfrak{x},\tau^*))\right]\right]\right) \tag{16}$$

which is the coupling of the Kamal transform and the homotopy perturbation strategy utilizing He's polynomials .Comparing the coefficient of like powers of $\mathcal{P}$, the accompanying approximations are acquired

$\mathcal{P}^0\ :\omega_0(\mathfrak{x},\tau^*) = \mathbb{E}(\mathfrak{x},\tau^*),$

$$\mathcal{P}^1 : \omega_1(\mathfrak{x}, \tau^*) = -\left[\mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}[\mathcal{H}_0'(\omega) + \mathrm{T}\omega_0(\mathfrak{x}, \tau^*)]\right]\right], \tag{17}$$

$$\mathcal{P}^2 : \omega_2(\mathfrak{x}, \tau^*) = -\left[\mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}[\mathcal{H}_1'(\omega) + \mathrm{T}\omega_1(\mathfrak{x}, \tau^*)]\right]\right],$$

$$\mathcal{P}^3 : \omega_3(\mathfrak{x}, \tau^*) = -\left[\mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}[\mathcal{H}_2'(\omega) + \mathrm{T}\omega_2(\mathfrak{x}, \tau^*)]\right]\right],$$

$$\dots\dots$$

$$\mathcal{P}^n : \omega_n(\mathfrak{x}, \tau^*) = -\left[\mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}[\mathcal{H}_{n-1}'(\omega) + \mathrm{T}\omega_{n-1}(\mathfrak{x}, \tau^*)]\right]\right],$$

$$\mathcal{P}^{n+1} : \omega_{n+1}(\mathfrak{x}, \tau^*) = -\left[\mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}[\mathcal{H}_n'(\omega) + \mathrm{T}\omega_n(\mathfrak{x}, \tau^*)]\right]\right].$$

Then the solution is:

$$\omega(\mathfrak{x}, \tau^*) = \omega_0(\mathfrak{x}, \tau^*) + \omega_1(\mathfrak{x}, \tau^*) + \omega_2(\mathfrak{x}, \tau^*) + \omega_3(\mathfrak{x}, \tau^*) + \cdots + \omega_n(\mathfrak{x}, \tau^*) + \cdots.$$

Example 1. Consider the inhomogeneous Advection problem [11]

$$\frac{\partial\omega}{\partial\tau^*} + \omega\frac{\partial\omega}{\partial\tau^*} = -\sin(\mathfrak{x} + \tau^*) - \frac{1}{2}\sin 2(\mathfrak{x} + \tau^*),\ \ \omega(\mathfrak{x}, 0) = \cos\mathfrak{x} \tag{18}$$

Standard HPM: According to homotopy Equation (3) we have

$$\frac{\partial v^\star}{\partial\tau^*} - \frac{\partial\omega_0}{\partial\tau^*} + \mathcal{P}\left(v^\star\frac{\partial v}{\partial\mathfrak{x}} + \frac{\partial\omega_0}{\partial\tau^*} + \sin(\mathfrak{x} + \tau^*) + \frac{1}{2}\sin 2(\mathfrak{x} + \tau^*)\right) = 0 \tag{19}$$

and the solution for first few step reads.

$$v_0^\star = \cos\mathfrak{x},$$

$$v_1^\star = \frac{1}{2}\tau^*\sin 2\mathfrak{x} + \cos(\mathfrak{x} + \tau^*) - \cos\mathfrak{x} + \frac{1}{4}\cos 2(\mathfrak{x} + \tau^*) - \frac{1}{4}\cos 2\mathfrak{x},$$

$$v_2^\star = -\frac{1}{4}\tau^{*2}\sin x\sin 2\mathfrak{x} + \frac{1}{2}\tau^{*2}\cos\mathfrak{x}\cos 2\mathfrak{x} - \sin\mathfrak{x}\sin(\mathfrak{x} + \tau^*) + \sin^2\mathfrak{x}$$

$$+\cos\mathfrak{x}\cos(\mathfrak{x} + \tau^*) + \cos^2\mathfrak{x} + \tau^*\sin 2\mathfrak{x} - \frac{1}{8}\sin\mathfrak{x}\sin 2(\mathfrak{x} + \tau^*) + \frac{1}{8}\sin\mathfrak{x}\sin 2\mathfrak{x}$$

$$+\frac{1}{4}\cos\mathfrak{x}\cos 2(\mathfrak{x} + \tau^*) - \frac{1}{4}\cos\mathfrak{x}\cos 2\mathfrak{x} + \frac{1}{4}\tau^*\sin\mathfrak{x}\cos 2\mathfrak{x}\frac{1}{2}\tau^*\cos\mathfrak{x}\sin 2\mathfrak{x}$$

$$\dots$$

Therefore, the approximate solution of equation (19) can be written as

$$\omega = \frac{1}{16}(\cos\mathfrak{x} - 2\tau^{*2}\cos\mathfrak{x} + 12\cos 2\mathfrak{x} + 3\cos 3\mathfrak{x} - 6\tau^{*2}\cos 3\mathfrak{x} + 16\cos(\mathfrak{x} + \tau^*) - \cos(2\tau^* + \mathfrak{x}) + 16\cos(2\mathfrak{x} + 4) + 4\cos(2\mathfrak{x} + \tau^*) - 3\cos(3\mathfrak{x} + \tau^*) - 2\tau^*\sin\mathfrak{x} - 8\tau^*\sin 2\mathfrak{x} - 6\tau^*\sin 3\mathfrak{x}) + \dots \tag{20}$$

HPKTM: to solve Eq.(18) by taking the Kamal transform on the both sides, subject to the initial condition, we get

$$\mathbb{E}(\mathfrak{x}, \ell) = \ell\cos\mathfrak{x} + \ell\left[\left(\frac{-\ell\sin\mathfrak{x} - \ell^2\cos\mathfrak{x}}{1+\ell^2}\right) - \frac{1}{2}\left(\frac{2\ell^2\cos\mathfrak{x} + \ell\sin 2\mathfrak{x}}{1+4\ell^2}\right)\right] - \ell\mathfrak{K}\left[\omega\frac{\partial\omega}{\partial\tau^*}\right] \tag{21}$$

Taking inverse Kamal transform, we get

$$\omega(\mathfrak{x}, \tau^*) = \cos(\mathfrak{x} + \tau^*) + \frac{1}{4}\cos 2(\mathfrak{x} + \tau^*) - \frac{1}{4}\cos 2\mathfrak{x} - \mathfrak{K}^{-1}\left[\ell\mathfrak{K}\left(\omega\frac{\partial\omega}{\partial\mathfrak{x}}\right)\right] \tag{22}$$

Now we apply the homotopy perturbation method; we have

$$\omega(\mathfrak{x},\tau^*) = \sum_{n=0}^{\infty}\mathcal{P}^n\omega_n(\mathfrak{x},\tau^*) \quad (23)$$

$$\sum_{n=0}^{\infty}\mathcal{P}^n\omega_n(\mathfrak{x},\tau^*) = \cos(\mathfrak{x}+\tau^*)+\frac{1}{4}\cos 2(\mathfrak{x}+\tau^*)-\frac{1}{4}\cos 2\mathfrak{x}-\mathcal{P}[\mathfrak{K}^{-1}\{\ell\mathfrak{K}(\sum_{n=0}^{\infty}\mathcal{P}^n\mathcal{H}_0')\}] \quad (24)$$

Where $\mathcal{H}_n'$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, for example, are given by

$$\mathcal{H}_0'(\omega) = \omega_0\omega_{0\mathfrak{x}},$$

$$\mathcal{H}_1'(\omega) = \omega_0\omega_{1\mathfrak{x}}+\omega_1\omega_{0\mathfrak{x}},$$

Comparing the coefficient of like powers of $\mathcal{P}$, we have

$$\mathcal{P}^0 : \omega_0(\mathfrak{x},\tau^*) = \cos(\mathfrak{x}+\tau^*)+\frac{1}{4}\cos 2(\mathfrak{x}+\tau^*)-\frac{1}{4}\cos 2\mathfrak{x},$$

$$\mathcal{P}^1 : \omega_1(\mathfrak{x},\tau^*) = -\mathfrak{K}^{-1}\left[\ell\mathfrak{K}\left(\mathcal{H}_0'(\omega)\right)\right] = -\frac{1}{4}\cos 2(\mathfrak{x}+\tau^*)+\frac{1}{4}\cos\mathfrak{x}+\frac{1}{64}\cos 4\mathfrak{x}+\cdots,$$

It is important to recall here that the noise terms appear between the components $\omega_0(\mathfrak{x},\tau^*)$ and $\omega_1(\mathfrak{x},\tau^*)$, more precisely, the noise terms $\pm\frac{1}{4}\cos 2(\mathfrak{x}+\tau^*)\pm\frac{1}{4}\cos 2\mathfrak{x}$ between the components $\omega_0(\mathfrak{x},\tau^*)$ and $\omega_1(\mathfrak{x},\tau^*)$ can be cancelled and remaining terms of $\omega_0(\mathfrak{x},\tau^*)$ still satisfy the equation. The exact solution is therefore

$$\omega(\mathfrak{x},\tau^*) = \cos(\mathfrak{x}+\tau^*) \quad (25)$$

Remark: The comparison of results between Laplace transform coupled with HPM [1] are same.

Example 2. Consider the inhomogeneous non-liner Klein Gordon equation [12]

$$\frac{\partial^2\omega}{\partial\tau^{*\,2}}-\frac{\partial^2\omega}{\partial\mathfrak{x}^2}+\omega^2 = -\mathfrak{x}\cos\tau^*+\mathfrak{x}^2\cos^2\tau^* \quad (26)$$

Subject to the initial condition $\omega(\mathfrak{x},0) = \mathfrak{x}$, $\frac{\partial\omega}{\partial\tau^*}(\mathfrak{x},0) = 0$ (27)

Standard HPM; according to homotopy Eq.(3) we have

$$\frac{\partial v^{\star 2}}{\partial\tau^{*\,2}}-\frac{\partial^2\omega_0}{\partial\tau^{*\,2}}+\mathcal{P}\left(\frac{\partial^2\omega_0}{\partial\mathfrak{x}^2}-\frac{\partial^2 v^\star}{\partial\mathfrak{x}^2}+v^{\star 2}+\mathfrak{x}\cos\tau^*-\mathfrak{x}^2\cos^2\tau^*\right) = 0 \quad (28)$$

And the solution for first few steps reads:

$$v_0^\star = \mathfrak{x}$$

$$v_1^\star = -\mathfrak{x}+\frac{1}{8}\mathfrak{x}^2-\frac{3}{4}\mathfrak{x}^2\tau^{*2}+\mathfrak{x}\cos\tau^*-\frac{1}{8}\mathfrak{x}^2\cos 2\mathfrak{x}$$

$$v_2^\star = -\frac{1}{16}\mathfrak{x}^2+\frac{1}{8}\tau^{*2}-\frac{\tau^{*2}}{24}-2\mathfrak{x}^2+\tau^{*2}\mathfrak{x}^2+\frac{\mathfrak{x}^3}{16}-\frac{\tau^{*2}\mathfrak{x}^3}{8}+\frac{\tau^{*4}\mathfrak{x}^3}{24}+2\mathfrak{x}^2\cos\tau^* -\frac{1}{16}\cos 2\mathfrak{x}-\frac{1}{16}\mathfrak{x}^3\cos 2\mathfrak{x}$$

Therefore, the approximate solution of Eq.(26) with condition Eq.(27)can be written as

$$\omega(\mathfrak{x},\tau^*) = -\frac{31}{16}\mathfrak{x}^2 - \frac{3}{4}\mathfrak{x}^2\tau^{*2} + \frac{1}{8}\tau^{*2} - \frac{\tau^{*4}}{24} + \tau^{*2}\mathfrak{x}^2 + \frac{\mathfrak{x}^3}{16} - \frac{\tau^{*2}\mathfrak{x}^3}{8} + \frac{\tau^{*4}\mathfrak{x}^3}{24} + \mathfrak{x}\cos\tau^*$$

$$\frac{1}{8}\mathfrak{x}^2\cos 2\mathfrak{x} + 2\mathfrak{x}^2\cos\tau^* - \frac{1}{16}\cos 2\mathfrak{x} - \frac{1}{16}\mathfrak{x}^3\cos 2\mathfrak{x} + \cdots \qquad (29)$$

HPKTM: To solve Eq.(26) by taking the kamala transform on the both sides, subject to the initial condition, we get

$$\mathbb{E}(\mathfrak{x},\ell) = \mathfrak{x}\ell - \frac{\mathfrak{x}\ell^3}{1+\ell^2} + \mathfrak{x}^2\ell^3\left[\frac{1+2\ell^2}{1+g\ell^2}\right] + \ell^2\mathfrak{K}\left[\frac{\partial^2\omega}{\partial\mathfrak{x}^2} - \omega^2\right] \qquad (30)$$

Tanking inverse Kamaal transform, we get

$$\omega(\mathfrak{x},\tau^*) = \mathfrak{x}\cos\tau^* - \frac{1}{8}\mathfrak{x}^2\cos\tau^* + \frac{1}{4}\mathfrak{x}^2\tau^{*2} + \frac{1}{8}\mathfrak{x}^2 + \mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}\left(\frac{\partial\omega^2}{\partial\mathfrak{x}^2} - \omega^2\right)\right] \qquad (31)$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty}\mathcal{P}^n\omega_n(\mathfrak{x},\tau^*) = \mathfrak{x}\cos\tau^* - \frac{1}{8}\mathfrak{x}^2\cos\tau^* + \frac{1}{4}\mathfrak{x}^2\tau^{*2} + \frac{1}{8}\mathfrak{x}^2$$

$$+\mathcal{P}\left[\mathfrak{K}^{-1}\left\{\ell^2\mathfrak{K}\left(\frac{\partial^2}{\partial\mathfrak{x}^2}\sum_{n=0}^{\infty}\mathcal{P}^n\omega_n(\mathfrak{x},\tau^*) - \sum_{n=0}^{\infty}\mathcal{P}^n\mathcal{H}_n'\right)\right\}\right] \qquad (32)$$

The first few components of He's polynomials, for example, for given by

$$\mathcal{H}_0'(\omega) = \omega_0^2,$$

$$\mathcal{H}_1'(\omega) = 2\omega_0\omega_1,$$

Comparing the coefficient of like powers of $\mathcal{P}$, we have

$$\mathcal{P}^0: \omega_0(\mathfrak{x},\tau^*) = \mathfrak{x}\cos\tau^* - \frac{1}{8}\mathfrak{x}^2\cos\tau^* + \frac{1}{4}\mathfrak{x}^2\tau^{*2} + \frac{1}{8}\mathfrak{x}^2,$$

$$\mathcal{P}^1: \omega_1(\mathfrak{x},\tau^*) = \mathfrak{K}^{-1}\left[\ell\mathfrak{K}\left(\frac{\partial^2\omega}{\partial\mathfrak{x}^2} - \mathcal{H}_0'(\omega)\right)\right] = \frac{1}{8}\mathfrak{x}^2\cos\tau^* - \frac{1}{4}\mathfrak{x}^2\tau^{*2} - \frac{1}{8}\mathfrak{x}^2 + \frac{1}{64}\mathfrak{x}^4\cos 2\tau^* + \cdots$$

The noise terms $\pm\frac{1}{8}\mathfrak{x}^2\cos\tau^* \pm\frac{1}{4}\mathfrak{x}^2\tau^{*2} \pm\frac{1}{8}\mathfrak{x}^2$ between the components $\omega_0(\mathfrak{x},\tau^*)$ and $\omega_1(\mathfrak{x},\tau^*)$ can be cancelled and the remaining terms of $\omega_0(\mathfrak{x},\tau^*)$ still satisfy the equation. The exact solution is therefore

$$\omega(\mathfrak{x},\tau^*) = \mathfrak{x}\cos\tau^* \qquad (33)$$

Remark: The comparison of results between Laplace transform coupled with HPM [1] are same.

Example 3. Consider the following non homogeneous nonlinear PDE[13]

$$\frac{\partial^2\omega}{\partial\tau^{*2}} + \frac{\partial^2\omega}{\partial\mathfrak{x}^2} + \left(\frac{\partial\omega}{\partial\mathfrak{x}}\right)^2 = 2\mathfrak{x} + \tau^{*2}, \qquad (34)$$

With the initial condition

$$\omega(\mathfrak{x},0) = 0, \frac{\partial}{\partial\tau^*}\omega(\mathfrak{x},0) = \mathrm{a}, \tag{35}$$

Standard HPM: According to homotopy perturbation method we have:

$$\frac{\partial \mathrm{v}^{\star 2}}{\partial \tau^{*2}} - \frac{\partial^2 \omega_0}{\partial \tau^{*2}} + \mathcal{P}\left(\frac{\partial^2 \mathrm{v}^\star}{\partial \mathfrak{x}^2} + \left(\frac{\partial \mathrm{v}^\star}{\partial \mathfrak{x}}\right)^2 + \frac{\partial^2 \omega_0}{\partial \mathfrak{x}^2} - 2\mathfrak{x} + \tau^{*2}\right) = 0 \tag{36}$$

Let's ignore the first few steps and start from determining $\mathrm{v}_i^\star$

$$\mathrm{v}_0^\star = \mathrm{a}\tau^*,$$

$$\mathrm{v}_1^\star = \mathfrak{x}\tau^{*2} + \frac{1}{30}\tau^{*6},$$

$$\mathrm{v}_2^\star = 0,$$

$$\mathrm{v}_3^\star = -\frac{1}{30}\tau^{*6},$$

$$\mathrm{v}_\mathrm{k}^\star = 0, \quad \mathrm{k} \geq 4$$

Therefore, we obtain

$$\mathrm{v}_0^\star = \mathrm{v}_0^\star + \mathrm{v}_1^\star + \mathrm{v}_2^\star + \mathrm{v}_3^\star + \mathrm{v}_4^\star + \cdots \mathrm{v}_\mathrm{k}^\star = \mathrm{a}\tau^* + \mathfrak{x}\tau^{*2}. \tag{37}$$

HPKTM: To solve Eq. (34) by taking the Kamal transform on the both sides, subject to the initial condition, we get

$$\mathbb{E}(\mathfrak{x},\ell) = \mathrm{a}\ell^2 + 2\mathfrak{x}\ell^3 + 4!\,\ell^7 - \ell^2\mathfrak{K}\left[\frac{\partial^2\omega}{\partial \mathfrak{x}^2} + \left(\frac{\partial\omega}{\partial \mathfrak{x}}\right)^2\right] \tag{38}$$

Taking inverse Kamal transform, we get

$$\omega(\mathfrak{x},\tau^*) = \mathrm{a}\tau^* + \mathfrak{x}\tau^{*2} + \frac{1}{30}\tau^{*6} - \mathfrak{K}^{-1}\left[\ell^2\mathfrak{K}\left(\frac{\partial^2\omega}{\partial \mathfrak{x}^2} + \left(\frac{\partial\omega}{\partial \mathfrak{x}}\right)^2\right)\right] \tag{39}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{\mathrm{n}=0}^{\infty}\mathcal{P}^\mathrm{n}\omega_\mathrm{n}(\mathfrak{x},\tau^*) = \mathrm{a}\tau^* + \mathfrak{x}\tau^{*2} + \frac{1}{30}\tau^{*6} - \mathcal{P}\left[\mathfrak{K}^{-1}\left\{\ell^2\mathfrak{K}\left(\frac{\partial^2\omega}{\partial \mathfrak{x}^2} + \sum_{\mathrm{n}=0}^{\infty}\mathcal{P}\mathcal{H}_\mathrm{n}'(\omega)\right)\right\}\right] \tag{40}$$

The first few components of He's polynomials, for example, for given by

$$\mathcal{H}_0'(\omega) = \left(\frac{\partial\omega_0}{\partial \mathfrak{x}}\right)^2 = \tau^{*4},$$

$$\mathcal{H}_1'(\omega) = 2\frac{\partial\omega_0}{\partial \mathfrak{x}}\frac{\partial\omega_1}{\partial \mathfrak{x}} = 0,$$

$$\mathcal{H}_2'(\omega) = \left(\frac{\partial\omega_1}{\partial \mathfrak{x}}\right)^2 + 2\frac{\partial\omega_0}{\partial \mathfrak{x}}\frac{\partial\omega_2}{\partial \mathfrak{x}} = 0,$$

Comparing the coefficient of like powers of $\mathcal{P}$, we have

$$\mathcal{P}^0: \omega_0(\mathfrak{x},\tau^*) = \mathrm{a}\tau^* + \mathfrak{x}\tau^{*2} + \frac{1}{30}\tau^{*6},$$

$$\mathcal{P}^1: \omega_1(\mathfrak{x}, \tau^*) = -\mathfrak{K}^{-1}\left[\ell^2\left(\mathfrak{K}\left(\frac{\partial^2\omega_0}{\partial \mathfrak{x}^2}\right) + \mathfrak{K}\left(\mathcal{H}_0'(\omega)\right)\right)\right] = -\frac{1}{30}\tau^{*6}$$

$$\mathcal{P}^2: \omega_2(\mathfrak{x}, \tau^*) = \mathfrak{K}^{-1}\left[\ell\mathfrak{K}\left(\left(\frac{\partial^2\omega_1}{\partial \mathfrak{x}^2}\right) + \mathfrak{K}\left(\mathcal{H}_1'(\omega)\right)\right)\right] = 0,$$

$$\dots\dots$$

$$\omega_k(\mathfrak{x}, \tau^*) = 0, \qquad k \geq 2$$

Therefore, the exact solution is given by

$$\omega(\mathfrak{x}, \tau^*) = a\tau^* + \mathfrak{x}\tau^{*2} \tag{41}$$

Remark: The comparison of results between Laplace transform coupled with HPM [1] are same.

Conclusions: Kamal homotopy perturbation s method has been illustrated. The arrangement of nonlinear partial differential issues by combination of homotopy perturbation method and Kamal transform is presented. This method has been effectively utilized to acquire the estimated solution of nonlinear partial differential issue. The outcome affirms that the Kamal transform method is a basic and great instrument. The results we got was similar to Laplace transform coupled with Homotopy Perturbation Method (HPTM)[1] .The results of both methods were same. It is in this manner demonstrated that the legitimacy of Homotopy Perturbation Kamal Transform (HPKTM) is dependable.

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