ON NUMBER OF UNITARY DIVISORS UNITARILY DIVISIBLE BY PRIME-POWER

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Abstract : A divisor d of n is called unitary divisor if d and n/d are relatively prime. A unitary divisor d of n is said to be unitarily divisible by p^{α} if it is divisible by p^{α} but not by $p^{\alpha+1}$. In this paper, asymptotic formula for the summatory function $T^{\#*}_{p^{\alpha}}(x)=\sum_{n\le x} t^{\#*}_{p^{\alpha}}(n)$ is derived where $t^{\#*}_{p^{\alpha}}(n)$ denotes the number of unitary divisors of n that are unitarily divisible by p^{α}.

IndexTerms **- Unitary divisor, asymptotic formula.**

I. Introduction

Let $t(n)$ denote the number of unitary divisors of n. Let $T(x)=\sum_{n\le x} t(n)$

In 1874, Mertens([3]) proved that

$$T(x)=\frac{1}{\zeta(2)}x\log x+\frac{1}{\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}\right)x+O\left(\sqrt{x}\log x\right) \qquad \ldots(1)$$

Where ζ is classical Riemann zeta function, ζ' is its derivative and γ is Euler's constant.

Eckford Cohen([1]) gave a different proof of (1). Gioia and Vaidya ([2]) improved the error-term to $O\left(\sqrt{x}\right)$. That is,

$$T(x)=\frac{1}{\zeta(2)}x\log x+\frac{1}{\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}\right)x+O\left(\sqrt{x}\right) \qquad \ldots(2)$$

Throughout this paper p is any fixed prime and α is any fixed positive integer.

In this paper, we find asymptotic formula for $T^{\#*}_{p^{\alpha}}(x)=\sum_{n\le x} t^{\#*}_{p^{\alpha}}(n)$, where $t^{\#*}_{p^{\alpha}}(n)$ denote the number of unitary divisor of n which are unitarily divisible by p^{α}, using asymptotic formula derived by Modi and Trivedi([5])

In the next section, we shall state the main theorem which we will prove in Section-4. Section-3 is devoted to supporting preliminary results. In the last section, one simple but interesting consequence of the main theorem is derived.

II. MAIN THEOREM

$$T^{\#*}_{p^{\alpha}}(x)=\sum_{n\le x} t^{\#*}_{p^{\alpha}}(n)=$$

$$\frac{p-1}{p^{\alpha}(p+1)\zeta(2)}x\log x+\frac{1}{p^{\alpha}(p+1)\zeta(2)}\left(\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}\right)(p-1)+\frac{(\alpha+2p-\alpha p^{2})\log p}{p+1}\right)+\mathrm{O}\left(\sqrt{x}\right)$$

III. PRELIMINARIES

Lemma 3.1. (Modi and Trivedi[5], p-117, Main Theorem)

If $t^{\#}_{p^{\alpha}}(n)$ denote the number of unitary divisor of n which are divisible by p^{α} then

$$T^{\#}_{p^{\alpha}}(x)=\sum_{n\le x} t^{\#}_{p^{\alpha}}(n)=\frac{1}{p^{\alpha-1}(p+1)\zeta(2)}x\log x+\frac{1}{p^{\alpha-1}(p+1)\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}-\frac{(p\alpha+\alpha-1)\log p}{p+1}\right)x+\mathrm{O}\left(\sqrt{x}\right).$$

Lemma 3.2.

$$T^{\#}_{p^{\alpha+1}}(x)=\sum_{n\le x}t^{\#}_{p^{\alpha+1}}(n)=\frac{1}{p^{\alpha}(p+1)\zeta(2)}x\log x+\frac{1}{p^{\alpha}(p+1)\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}-\frac{(p\alpha+p+\alpha)\log p}{p+1}\right)x+O\left(\sqrt{x}\right).$$

Proof. It is mere replacement of α by $\alpha+1$ in previous lemma.

Lemma 3.3.

For any natural number n, $t^{\#*}_{p^{\alpha}}(n)=t^{\#*}_{p^{\alpha}}(n)-t^{\#*}_{p^{\alpha+1}}(n)$.

Proof.

$$t^{\#*}_{p^{\alpha}}(n)$$

$$=\#\left\{d\,\|\,n : p^{\alpha}\mid d,\ p^{\alpha+1}\nmid d\right\}$$

$$=\#\left\{d\,\|\,n : p^{\alpha}\mid d\right\}-\#\left\{d\,\|\,n : p^{\alpha+1}\mid d\right\}$$

$$=t^{\#}_{p^{\alpha}}(n)-t^{\#}_{p^{\alpha+1}}(n)$$

IV. PROOF OF MAIN THEOREM

By Lemma 3.3,

$$T^{\#*}_{p^{\alpha}}(x)=\sum_{n\le x}t^{\#*}_{p^{\alpha}}(n)=\sum_{n\le x}\left(t^{\#}_{p^{\alpha}}(n)-t^{\#}_{p^{\alpha+1}}(n)\right)=\sum_{n\le x}t^{\#}_{p^{\alpha}}(n)-\sum_{n\le x}t^{\#}_{p^{\alpha+1}}(n)=T^{\#}_{p^{\alpha}}(x)-T^{\#}_{p^{\alpha+1}}(x)$$

Using Lemma 3.1 and Lemma 3.2,

$$T^{\#*}_{p^{\alpha}}(x)=T^{\#}_{p^{\alpha}}(x)-T^{\#}_{p^{\alpha+1}}(x)$$

$$=\left(\frac{1}{p^{\alpha-1}(p+1)\zeta(2)}x\log x+\frac{1}{p^{\alpha-1}(p+1)\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}-\frac{(p\alpha+\alpha-1)\log p}{p+1}\right)x+O\left(\sqrt{x}\right)\right)$$

$$-\left(\frac{1}{p^{\alpha}(p+1)\zeta(2)}x\log x+\frac{1}{p^{\alpha}(p+1)\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}-\frac{(p\alpha+p+\alpha)\log p}{p+1}\right)x+O\left(\sqrt{x}\right)\right)$$

$$=\frac{p-1}{p^{\alpha}(p+1)\zeta(2)}x\log x+\frac{1}{p^{\alpha}(p+1)\zeta(2)}\left(\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}\right)(p-1)+\frac{(\alpha+2p-\alpha p^{2})\log p}{p+1}\right)+O\left(\sqrt{x}\right)$$

Hence, the Theorem is established.

V. CONSEQUENCE OF MAIN THEOREM

Corollary 5.1.

If $t_{4,2}(n)$ denotes the number of unitary divisors of n which are of the form $4k+2$, then

$$T_{4,2}(x)=\sum_{n\le x}t_{4,2}(n)=\frac{1}{6\zeta(2)}x\log x+\frac{1}{6\zeta(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}+\frac{\log 2}{3}\right)+O\left(\sqrt{x}\right)$$

Proof. It can be seen that numbers of the form $4k+2$ are numbers unitarily divisible by 2 i.e. divisible by 2 but not by 4. Therefore, $t_{4,2}(n)=t^{\#*}_{2^{1}}(n)$ and consequently, $T_{4,2}(x)=T^{\#*}_{2^{1}}(x)$. Taking $p=2$ and $\alpha=1$ in the Main Theorem, we get the desired result.

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