RATIONAL POINTS ON CUBIC SURFACES

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ABSTRACT

A cubic surface is a projective variety studied in algebraic geometry. It is an algebraic surface in three-dimensional projective space. Cubic surfaces defined over the field of rational numbers often contain infinitely many rational points. In this present paper, we discuss the rational points on cubic surfaces. No method is known for determining whether rational points exist on a general cubic surface $F(x, y, z) = 0$, or for finding all of them if any exist. Geometric considerations may help in finding infinite solutions and even all the solutions for cubic surfaces.

Keywords: Diophantine equation, cubic surface, cubic polynomial, rational point, rational coefficient, rational number.

1. Introduction

Let $F(x, y, z)$ be a cubic polynomial with rational coefficient, so that

$$F(x, y, z) = 0 \quad (1)$$

represents a rational cubic surface in Cartesian 3-space. The problem of finding rational solutions x, y, z of equation (1) can be stated as that of determining the rational points lying on the rational surface F represented by equation (1). No method is known for determining whether rational points exist on a general cubic surface represented by equation (1), or for finding all of them if any exist. When one rational point $P(x_0, y_0, z_0)$ of equation (1) is known infinity of them can be found by geometric considerations which we proceed to discuss below.

2. Some definitions

F is said to be singular if there are some points $P(x, y, z)$ which satisfy $F = 0$ and

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0. \quad (2)$$

Such a point P is called a double point of F if all the partial derivatives of the second order of F do not vanish at P, otherwise P is called a triple point of F. A point $P(x, y, z)$ lying on F that is which satisfies $F = 0$ is called a simple point of F if it does not satisfy equation (2).

3. The Diophantine equation (1)

The equation (1) can be written in the form

$$\phi_0 + \phi_1(x, y\ z) + \phi_2(x, y\ z) + \phi_3(x, y\ z) = 0 \quad (3)$$

where ϕ_0 is a rational number and $\phi_i(i = 1, 2, 3)$ are homogeneous rational polynomials in x, y, z of degrees 1, 2, 3 respectively, the first two of which may possibly vanish identically.

Let us suppose that F is irreducible and that a particular rational solution (x_0, y_0, z_0) is known. By choice of the rational co-ordinates (x, y, z), the rational point $P(x_0, y_0, z_0)$ can be taken at the origin, so that from equation (3) $\phi_0 = 0$.

We distinguish three cases.

Case 3. Let P be a triple point.

If P is a triple point of F, then the equation (3) takes the form:

$$\phi_3(x, y, z) = 0$$

writing $x = \frac{x}{z}, \quad y = \frac{y}{z}, \quad z = \frac{x}{y},$ this can be written as

$$\phi_3(x, y, z) = 0 \quad (4)$$

The problem of finding rational points of F is now that of finding the rational points of a plane cubic curve equation (4). When equation (4) has obvious trivial solutions, initial rational points on ϕ_3 are located and then new solutions may be found by geometrical methods as follows.

Suppose that a line L intersects ϕ_3 two rational points P_1, P_2. Since ϕ_3 is of third degree, L must intersect it again in a third rational point P. For the equation of L is rational and the abscissa of P_1, P_2. P are obtained by solving the equations for ϕ_3 and L simultaneously. This is equivalent to finding the three roots of a rational cubic equation, and since two of them are known to be rational, so is the third. Thus given P_1, P_2 on ϕ_3 a new rational point P is located.

However, it may happen that P coincides with P_1 or P_2 or that P lies at infinity, in which case no new results.

Alternatively if P_1 is a rational point on ϕ_3, take L to be the tangent to ϕ_3 at P_1. L has a rational equation and double contact with ϕ_3 at P_1 and therefore a third rational intersection with ϕ_3 which may provide a new solution.

Similarly, if ϕ_3 has a rational asymptote A and P_1 is a rational point on ϕ_3, take L as the line through P_1 parallel to A. Then L has contact with ϕ_3 at P_1 and at infinity, and a third rational contact with ϕ_3 may give new solution.

We illustrate these geometric procedures for the equation

$$y^3 + y = ax^3 + a^3x \quad (5)$$

where a is a rational number.

We may interpret equation (5) as the cartesian equation of a cubic curve ϕ_3 in the xy-plane symmetric with respect to the origin and passing through the rational point P_0 with co-ordinates $x = 1$, $y = a$. Each rational point on ϕ_3 corresponds to a rational solution of equation (5). The tangent at $p_0(x_0, y_0)$ has slope

$$t = \frac{a^3 + 3ax_0^2}{3y_0^2 + 1}$$

and intersects ϕ_3 again in $P_1(x_1, y_1)$, where

$$x_1 = -2x_0 - \left[\frac{3t^2(y_0 - tx_0)}{(t^3 - a)}\right]$$

$$y_1 = -t(x_1 - x_0) + y_0$$

By setting $x_0 = 1$, $y_0 = a$, we find

$$x_1 = \frac{-2(4a^6 + a^4 + 10a^2 + 1)}{(a^2 + 1)(a^4 + 18a^2 + 1)}$$

$$y_1 = \frac{-2a(a^6 + 10a^4 + a^2 + 4)}{(a^2 + 1)(a^4 - 18a^2 + 1)}$$

giving a second rational solution to equation (5).

If $a = m^3$, ϕ_3 has the rational asymptote $y = mx$, and a line through $P_0(x_0, y_0)$ parallel to the asymptote must intersect ϕ_3 again in a rational $P_2(x_2, y_2)$. We find

$$x_2 = \frac{1 + (y_0 - mx_0)^2}{(2m^3 - x_0)}$$

$$y_1 = m(x_2 - x_0) + y_0$$

If $x_0 = 1$, $y_0 = a = m^3$, then we get

$$x_2 = \frac{(m^6 - 2m^4 + m^2 + 1)}{3m^2}$$

$$y_2 = \frac{m(m^6 + m^4 - m^2 + 1)}{3ma^2}$$

as a rational of equation (5) with $a = m^3$.

Again if two rational points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ of ϕ_3 are known then third one $P_3(x_3, y_3)$ can be readily obtained. The slops of the line $P_1 P_2$ is

$$t = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

and

$$x_3 = -\left[x_1 + x_2 + \frac{3t^2(y_1 - tx_1)}{(t^3 - a)}\right]$$

$$y_3 = t(x_3 - x_1) + y.$$

Case 2. Let P be a double point.

If P is a double point of F, then the equation (3) takes the form

$$\phi_2(x, y, z) + \phi_3(x, y, z) = 0 \qquad (6)$$

where ϕ_2 and ϕ_3 are two relatively prime polynomials, neither of which vanishes identically. There is now a one–one correspondence between the rational points Q and F and the rational lines PQ containing P, since the double point of P being rational, all the rational points on the surface equation (6) are given by the third intersection with surface of the line

$$\frac{x-0}{\ell} = \frac{y-0}{m} = \frac{z-0}{n}$$

i.e. $\dfrac{x}{\ell} = \dfrac{y}{\mu} = \dfrac{z}{I}$

where $\lambda = \dfrac{\ell}{n}, \mu = \dfrac{m}{n}$ are rational parameters. This gives the following complete two parameters solution of equation (6) degree three

$$x = \frac{-\lambda\phi_2(\lambda, \mu, \ell)}{\phi_3(\lambda, \mu, \ell)},$$

$$y = \frac{-\mu\phi_2(\lambda, \mu, \ell)}{\phi_3(\lambda, \mu, \ell)},$$

$$z = \frac{-\phi_2(\lambda, \mu, \ell)}{\phi_3(\lambda, \mu, \ell)}.$$

Case 3. Let P be a simple point

If P is a simple point of F, then in equation (3) we have $\phi_0 = 0$ but $\phi_1(x, y, z) \neq 0$. The equation $\phi_1(x, y, z) = 0$ represents the plane touching F at P, on taking this plane as the xy-plane, the equation of F takes the form

$z + \phi_2(x, y, z) + \phi_2(x, y, z) = 0$ (7)

We further distinguish two cases according as

$\phi_2(x, y, z) \neq 0$ or $\phi_2(x, y, z) = 0$

Let $\phi_2(x, y, z) \neq 0$ identically, then there is a one-one correspondence between the rational points Q and F lying on π and the rational lines PQ of π containing P. This gives the following complete one parameter solution of equation (7).

$$x = \frac{-\lambda\phi_2(\lambda, \mu, \ell)}{\phi_3(\lambda, \mu, \ell)},$$

$$y = \frac{-\phi_2(\lambda, \mu, \ell)}{\phi_3(\lambda, \mu, \ell)},$$

$z = 0.$

Let $\phi_2(x, y, z) = 0$ identically. Then the equation (7) can be written in the form

$$Z = [1 + \psi_1(x, y, z)] + \phi_3\left(\frac{x}{z}, \frac{y}{z}, 1\right) = 0$$

$$\text{or, } \left[\frac{1}{z} + \psi_1 \frac{\left(\frac{x}{z}, \frac{y}{z}, 1\right)}{2}\right]^2 - \frac{1}{4}\psi_1^2\left(\frac{x}{z}, \frac{y}{z}, 1\right) + \phi_3\left(\frac{x}{z}, \frac{y}{z}, 1\right) = 0$$

Hence on putting

$$X = \frac{x}{2}, Y = \frac{y}{2}, Z = \frac{1 + 2^{-1}\psi_1(x, y, z)}{z}$$

The equation (7) takes the form

$z^2 = f(X, Y)$ (8)

where f is a rational cubic polynomial in X, Y.

We conclude by giving the following method of solution of the general equation (3) essentially due to white head [1].

Let

$z^2 = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$ (9)

Where the coefficients are rational numbers.

We may suppose that $a \neq 0$ and then that $a = 1$on replace x, y by x/a, z/a. Again on replacing x by $x - py - q$, say, we may suppose that the equation takes the parameters

$$z^2 = x^3 + x(ay^2 + by + c) + dy^3 + ey^2 + fy + g.$$

The right hand side is a function of only one variable if and only if $a = b = d = e = f = 0$, a case we have excluded.

Put $x = y^2 + 2ty$ where t is a parameters. Then

$z^2 = y^6 + 6ty^5 + (12t^2 + a)y^4 + (8t^3 + 2at + b + d)y^3 + (2bt + c + e)y^2 + (2ct + f)\, y + g.$

Put now $z = y^3 + Ay^2 + By + C$, where

$2A = 6t$, $2B + A^2 = 12\, t^2 + a$, $2C + 2AB = 8t^2 + 2at + b + d.$

and so $A = 3t$, $2B = 3t^2 + a$, $2C = b + d - at - t^3$.

Substituting for z^2 we have

$PY^2 + Qy + R = 0$

where

$$P + \frac{1}{4}(3t^2 + a) + 3t(b + d - at - t^3) = 2bt + c + e,$$

$$\text{or, } P = \frac{3}{4}t^4 + \frac{3}{2}at^2 - (b + 3d)t + c + e - \frac{1}{4}a^2.$$

$$\text{So, } Q + \frac{1}{2}(3t^2 + a) + (b + d - at - t^3) = 2ct + f,$$

$$\text{or, } Q = \frac{3}{4}t^5 + 2at^3 - \frac{3}{2}(b + d)t^2 + \frac{1}{2}(a^2 + 4c)t + f - \frac{1}{2}a(b + d).$$

$$\text{Finally, } R = g - \frac{1}{4}(b + d - at^2 - t^3)^2 = \frac{1}{4t^6} + \ldots$$

We note that

$2Pt - Q = at^3 - \frac{1}{2}(b + 9d)t^2 + (2e - a^2)t + \frac{1}{2}a(b + d) - f$, and this is not identically zero unless $a = e = f = 0$, $b + 9d = 0$.

On solving the quadratic for y, we have two values of y which are finite and different, except for a finite number of values of t since neither P nor $\Delta = Q^2 - 4PR = 3t^{10} + \ldots$ is identically zero.

Hence $2Py = -Q \pm \sqrt{\Delta}$

$$2P^2x = 2P^2y^2 + 4P^4ty$$
$$= -2PQy - 2PR + 4P^4ty$$
$$= y(4P^4t - 2PQ) - 2PR$$
$$= (-Q \pm \sqrt{\Delta})(2Pt - Q) - 2PR$$
$$= Q^2 - 2PR - 2PQt \pm (2Pt - Q)\sqrt{\Delta}.$$

Since $z = y^3 + \ldots$, we have, say

$$x = \alpha \pm \alpha_1\sqrt{\Delta}, \quad y = \beta \pm \beta_1\sqrt{\Delta}, \quad z = \gamma \pm \gamma_1\sqrt{\Delta},$$

These two points define a straight line and any point on it can be written as

$$x = \alpha \pm \alpha_1\theta, \quad y = \beta \pm \beta_1\theta, \quad z = \gamma \pm \gamma_1\theta,$$

where θ is a parameter. The points where this line meets the cubic surface $Z^2 = f(X, y)$ are given by a cubic equation in θ. This has a factor $\theta^2 - \Delta$, and the remaining factor is linear, and determines a rational point. The third root will not be infinite except for a finite number of values of t. For if the coefficient of θ^3 is zero,

$$\alpha_1^3 + \alpha_1\beta_1^2 + d\beta_1^3 = 0$$

or, $(2Pt - Q)^3 + aP^2(2Pt - Q) + dP^3 = 0.$

On noting the degrees of $2Pt - Q$ and P and the coefficient of t^{12} in the equation we find $d = 0$. If $2Pt - Q$ is identically zero, $a = e = f = 0$, $b + 9d = 0$ i.e. $b = 0$ and this is the excluded case. If $2Pt - Q$ is not identically zero

$$(2Pt - Q)^3 + aP^2 = 0.$$

From the coefficient of t^8, $a = 0$, and then $2Pt - Q = 0$, a contradiction.

References

[1] Guy, R.K. (2004). "*Unsolved Problems in Number Theory*", 3rd ed., New York: Springer-Verlag.

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