INDEPENDENT DOMINATION IN SOME SUBDIVISION GRAPHS

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Abstract

A dominating set S of a subdivision graph S(G) = (V, E) is called an independent dominating set if the induced subgraph $\langle S \rangle$ has no edges. The independent domination number i[S(G)] of a graph S(G) is the minimum cardinality of an independent dominating set.

keywords : independent domination, independent domination number, subdivision graph

1. Introduction

In this paper by a graph G we mean G is a simple finite, undirected and connected graph without loops and multiple edges. The theory of independent domination was formalized by Berge and Ore in 1962. The independent domination number i(G) were introduced by Cockayne and Hedetniemi (1974,1972). The vertex and edge set of a graph G denoted by V(G) and E(G) respectively. The degree of any vertex u in G is the number of edges incident with u. The maximum degree of a graph G is denoted by Δ(G).

In this paper we investigate independent domination number of some subdivision graphs.

2. Preliminaries

Definition 2.1

A dominating set S of a subdivision graph S(G) = (V, E) is called an independent dominating set if the induced subgraph $\langle S \rangle$ has no edges. The independent domination number i[S(G)] of a graph S(G) is the minimum cardinality of an independent dominating set.

Definition 2.2

The subdivision graph S(G) of a graph G is that graph obtained from G by replacing each e = uv of G by a new vertex w and the two new edges uw and vw. Equivalently, each edge of G is replaced by a path of length 2.

Definition 2.3

The Barbell graph Bb_n is defined as the simple graph obtained by connecting two copies of a complete graph K_n by a bridge.

Definition 2.4

The Gear graph G_n is a wheel graph with graph vertex added between each pair of adjacent vertices of the outer cycle.

Definition 2.5

The product graph $P_n \times P_2$ is called a Ladder and it is denoted by L_n.

Definition 2.6

The graph obtained by attaching a pendant edge to both sides of each vertex of a path P_n. This graph is called Double Comb. It is denoted by D(C_n).

Definition 2.7

The graph $C_n \odot K_{1,2}$ is obtained by attaching $K_{1,2}$ to each vertex of C_n.

Definition 2.8

The Double Wheel graph DW_{2n+1} is defined as the graph $2C_{2n}+ K_1$ where K_1 is the singleton graph and C_n is the cycle graph.

Result 2.9

For the path and cycle, i(P_n) = i(C_n) = $\lceil n/3 \rceil$.

3. Main Results

Theorem 3.1

The independent domination number of the subdivision of Barbell graph Bb_n $(n \geq 4)$ is 2n – 2.

Proof:

The subdivision graph of Barbell graph Bb_n contains n (n + 1) + 1 vertices.

Among this 2n vertices are actual vertices and [n (n – 1) + 1] vertices are the subdivision vertices of the graph S(Bb_n).

The graph S(Bb_n) contain the subgraphs two copies of S(K_n) that is known as S(P_1).

Among 2n vertices, 2 (n – 1) vertices are of degree (n – 1) and the remaining 2 vertices are of degree n. The subdivision vertices of S(Bb_n) are of degree 2.

Label the vertices of S(Bb_n) as follows:

Let $u_1, u_2, \ldots \ldots \ldots, u_n$ be the vertices of the first copies of K_n and let v_i , $1 \leq i \leq \left\lfloor \frac{n(n-1)+1}{2} \right\rfloor$ be the subdivision vertices of first copies of K_n. Let c be the subdivision vertex of the bridge.

Let $u_1^1, u_2^1, \ldots \ldots \ldots, u_n^1$ be the vertices of second copies of K_n and let v_i^1, $1 \leq i \leq \left\lfloor \frac{n(n-1)+1}{2} \right\rfloor$ be the subdivision vertices of second copies of K_n.

Now to attain the minimum cardinality, every independent dominating set of S(Bb_n) must contain the vertices u_n and u_n^1 because deg (u_n) = n = deg (u_n^1). Vertices u_n and u_n^1 dominates n vertices of the first and second copies of S(K_n) which is also dominate the vertex c.

Now we consider n – 3 vertex of degree n – 1 of each copies of S(K_n) of S(Bb_n) . These vertices dominate the inner part of the subdivision vertices.

In order to dominate the remaining vertices of the outer part of each copies of S(K_n), one of the subdivision vertex of outer part of S(K_n) which is not adjacent to consider a n – 3 vertices of degree (n -1).

Hence, at least 2 [1 (vertex of degree n) + n – 3 (vertex of degree (n – 1) + 1 (subdivision vertex on the outer part of S(K_n)] = 2 (n – 3 + 2) = 2 (n – 1) vertices are required to dominate all the vertices of S(Bb_n). Moreover, the set S is also an independent set of S(Bb_n) because no two vertices in S are adjacent.

Therefore, for any independent dominating set S of S(Bb_n), $|S| \geq 2(n-1)$ implying that i[S(Bb_n)] = 2 (n – 1) = 2n – 2.

Hence, i[S(Bb_n)] = 2n – 2 for n ≥ 4.

Example 3.2

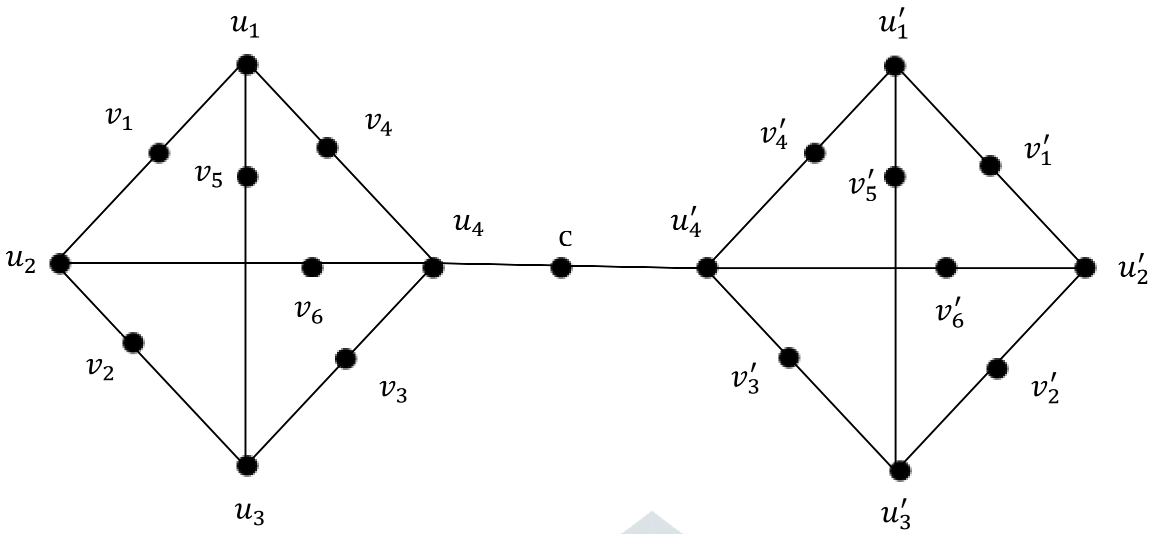

figure : 1

In figure 1, $\{u_1, u_4, v_2, u'_1, u'_4, v'_2\}$ is an independent dominating set with minimum cardinality. Thus, i[S(Bb_4)] = 6.

Theorem 3.3

The independent domination number of subdivision graph of Gear graph G_n (n ≥ 4) is $\lceil(4n-1)/3\rceil$.

Proof:

The subdivision graph of the Gear graph contains the subgraph S($C_{2(n-1)}$) and the hub of the wheel.

The subdivision graph of Gear graph S(G_n) has 5n – 4 vertices and 6 (n -1) edges.

The hub of the wheel is of degree n – 1. Among 4 (n – 1) vertices the n – 1 vertices are of degree 2. The subdivision vertices in the spoke of the wheel are of degree 2.

Let u_i, $1 \le i \le n-1$ be the rim vertices of the wheel. Let v_i, $1 \le i \le n-1$ be the middle vertices on the rim vertices of the wheel and w_i, $1 \le i \le n-1$ be the subdivision vertices of $C_{2(n-1)}$. Vertex w_i is adjacent to u_i and v_i for $1 \le i \le n-1$. Let w_i^1, $1 \le i \le n-1$ be the subdivision vertices of $C_{2(n-1)}$. Vertex w_i^1, is adjacent to v_i and u_{i+1} for $1 \le i \le n-2$.

Let c denote the apex vertex of wheel W_n. Let $c_1, c_2, \dots\dots\dots, c_{n-1}$ be the subdivision vertices in the spoke of the wheel. Vertex c_i is adjacent to c and u_i for $1 \le i \le n-1$.

Since deg (c) = Δ (S(G_n)) = n – 1 and the vertex c dominates the subdivision vertices in the spoke of the wheel. Thus, every independent dominating set of S(G_n) must contain the vertex c.

Now by Result 2.9, i (C_n) = $\lceil n/3\rceil$.

Therefore, at least $\lceil 4(n-1)/3\rceil$ non-adjacent vertices are required to dominate all the vertices of S($C_{2(n-1)}$) of S(G_n). Hence, at least

$\lceil 4(n-1)/3\rceil$ (vertices on S($C_{2(n-1)}$)) + 1 (hub of the wheel) = $\lceil 4n-1/3\rceil$ non-adjacent vertices are essential to dominate all the vertices of S(G_n).

Therefore, for any independent dominating set S of S(G_n), $|S| \ge \lceil 4n-1/3\rceil$, implying that

i[S(G_n)] = $\lceil 4n-1/3\rceil$.

For example i[S(G_6)] = $\{u_2, u_5, v_3, w_1, w_4, w_2^1, w_5^1$, c$\}$ as shown in the figure 2.

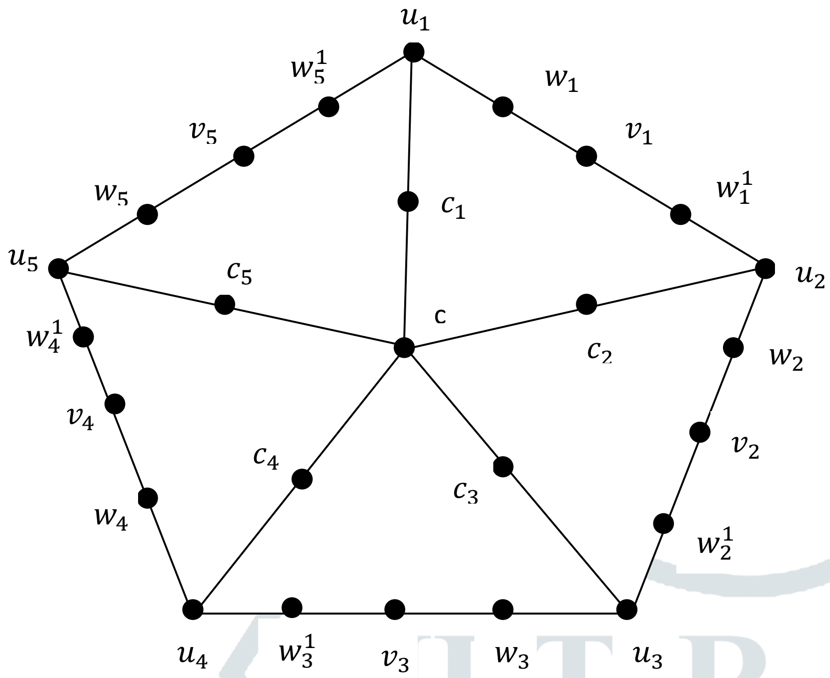

figure : 2

Theorem 3.4

The independent domination number of subdivision graph of Ladder graph L_n (n ≥ 3) is

$2\lceil(3n-2)/3\rceil$.

Proof:

For the subdivision graph of Ladder graph S (L_n)

$|V(S\,(L_n)|$ = 5n – 2 and $|E(S\,(L_n)|$ = 6n – 4.

Among this 2n vertices are actual vertices and 3n – 2 vertices are the subdivision vertices of the graph S (L_n).

Label the vertices of S (L_n) as follows:

Let u_i, $1 \le i \le n$ be the vertices on the upper part of the ladder and x_i, $1 \le i \le n-1$ be the subdivision vertices of upper part of L_n.

Let w_i, $1 \le i \le n$ be the subdivision vertices of middle part of the ladder.

Let v_i, $1 \le i \le n$ be the vertices of lower part of the ladder and y_i, $1 \le i \le n-1$ be the subdivision vertices of lower part of the ladder.

Vertex x_i, $1 \le i \le n-1$ is adjacent to u_i and u_{i+1} and y_i, $1 \le i \le n-1$ is adjacent to v_i and v_{i+1}. Vertex w_i, $1 \le i \le n$ is adjacent to u_i and v_i.

Now by Result 2.9, i (P_n) $= \lceil n/3\rceil$.

The graph S (L_n) contains two path with 2n – 1 vertices. Therefore, at least $2\lceil(2n-1)/3\rceil$ vertices are enough to dominate all the vertices of lower and upper part of S (L_n) which is also dominate the middle subdivision vertices which is adjacent to $2\lceil(2n-1)/3\rceil$ vertices on the path.

Now $2\lceil(n-1)/3\rceil$ subdivision vertices on the middle part of ladder are essential to dominate the remaining vertices of S (L_n). Hence, at least

$2\lceil(2n-1)/3\rceil + 2\lceil(n-1)/3\rceil = 2\lceil(2n-1+n-1)/3\rceil$

$= 2\lceil(3n-2)/3\rceil$

vertices are required to dominate all the vertices of S (L_n).

Since each vertex in S (L_n) is either in S or is adjacent to a vertex in S, it follows that the set S is a dominating set of S (L_n). Moreover, the set S is also an independent set of S (L_n) because no two vertices in S are adjacent. Therefore, the set S is an independent dominating set of S (L_n). As

$|S| = 2\lceil(3n-2)/3\rceil$

the set S is of minimum cardinality.

Hence, the set S is an independent dominating set with minimum cardinality implying that

i[S (L_n)] = $2\lceil(3n-2)/3\rceil$.

Example 3.5

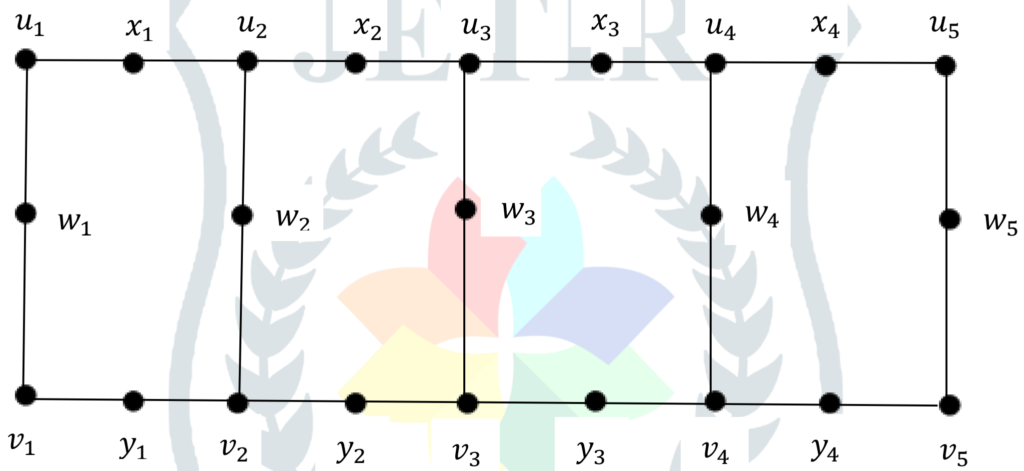

figure : 3

In figure 3, $\{x_1, x_4, u_3, v_3, y_1, y_4, w_1, w_2, w_4, w_5\}$ is the minimum independent dominating set. Hence, i[S (L_n)] = 10 for n = 5.

Theorem 3.6

The independent domination number of the subdivision graph of Double Comb D(C_n) is 3n – 2, n ≥ 3.

Proof:

For the subdivision graph of Double Comb S(D(C_n))

$|V$(S (D(C_n))| = 6n – 1 and $|E$(S (D(C_n))| = 6n – 2.

Among 6n – 1 vertices the 3n vertices are actual vertices and the remaining 3n – 1 vertices are the subdivision vertices.

The graph S(D(C_n)) contains the subgraph S(P_n) and S(K_1).

Let u_i, $1 \le i \le n$ be the vertices on the path P_n and v_i, $1 \le i \le n$ be upper path of the vertices. Let x_i, $1 \le i \le n$ be subdivision vertices of upper path of the vertices and u_i', $1 \le i \le n-1$ be the subdivision vertices of the path.

Let w_i, $1 \le i \le n$ be the lower path of vertices and y_i, $1 \le i \le n$ be the subdivision vertices of lower path of the vertices.

Vertex x_i, $1 \le i \le n$ is adjacent to u_i and v_i and u_i', $1 \le i \le n-1$ is adjacent to u_i and u_{i+1}. Vertex y_i, $1 \le i \le n$ is adjacent to u_i and w_i.

The graph S(D(C_n)), n – 2 vertices are of degree 4, 2n vertices are of degree1 and all the subdivision vertices are of degree 2.

Now to attain the minimum cardinality, every independent dominating set must contains n – 2 vertices of degree 4 and it contains the vertices x_1, y_1, x_n and y_n.

In order to dominate the remaining vertices of S(D(C_n)) at least 2 (n – 2) vertices are required.

Hence, at least n – 2 + 4 + 2 (n – 2) = 3n – 2 non-adjacent vertices are required to dominate all the vertices of S(D(C_n)).

Hence, for any independent dominating set S of S(D(C_n)), $|S|$ ≥ 3n – 2.

Thus, i[S(D(C_n))] = 3n – 2.

For example i[S(D(C_4))] = { $u_2, u_3, x_1, y_1, x_4, y_4, v_2, v_3, w_2, w_3$} as shown in the figure 4.

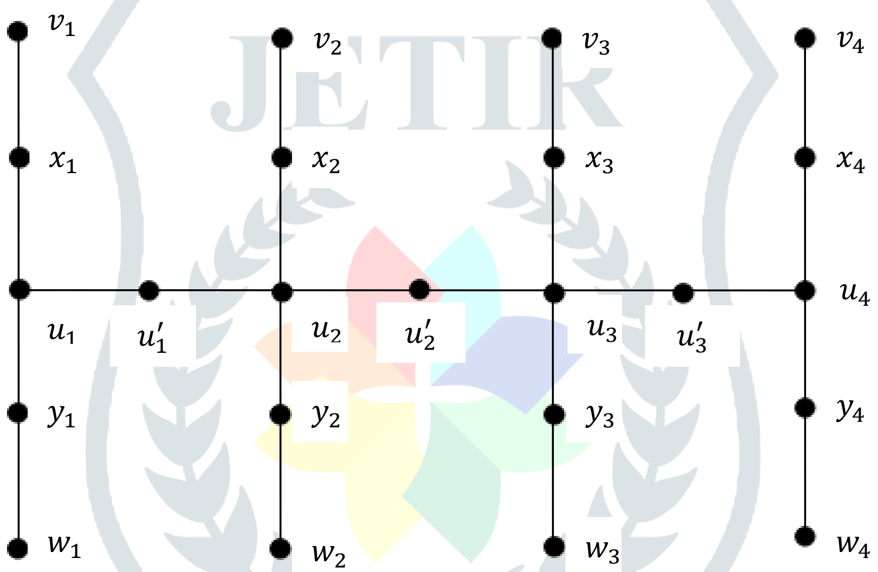

figure : 4

Theorem 3.7

For the subdivision graph of $C_n \odot K_{1,2}$ (n ≥ 3), i[S($C_n \odot K_{1,2}$)] = $\lceil 8n/3 \rceil$.

Proof:

The subdivision graph S($C_n \odot K_{1,2}$) has 6n vertices and 6n edges.

Among 6n vertices, 3n are actual vertices and the remaining 3n are subdivision vertices.

The graph S($C_n \odot K_{1,2}$) contains the subgraph S(C_n) and S($K_{1,2}$).

Let u_i, $1 \le i \le n$ be the vertices of cycle C_n and x_i, $1 \le i \le n$ be the subdivision vertices of the cycle. Let v_i, $1 \le i \le n$ be the vertices of $K_{1,2}$ and y_i, $1 \le i \le 2n$ be the subdivision vertices of $K_{1,2}$.

Since the graph S($C_n \odot K_{1,2}$) contains the cycle with 2n vertices.

Now by Result 2.9, i(C_{2n}) = $\lceil 2n/3 \rceil$. Therefore, at least $\lceil 2n/3 \rceil$ non-adjacent vertices are enough to dominate all the vertices of C_{2n} of S($C_n \odot K_{1,2}$).

In order to dominate the remaining vertices at least 2n vertices are required.

Hence, at least $\lceil 2n/3 \rceil$(vertices of $S(C_n)$) + 2n (vertices of S($K_{1,2}$) = $\lceil 8n/3 \rceil$ pairwise non-adjacent vertices are required to dominate all the vertices of $S(C_n \odot K_{1,2})$, $|S| \geq \lceil 8n/3 \rceil$.

Hence, $i[S(C_n \odot K_{1,2})] = \lceil 8n/3 \rceil$.

For example, $i[S(C_n \odot K_{1,2})] = \{u_3, x_1, x_4, y_1, y_2, y_3, y_4, y_7, y_8, v_5, v_6\}$ as shown in the figure 5.

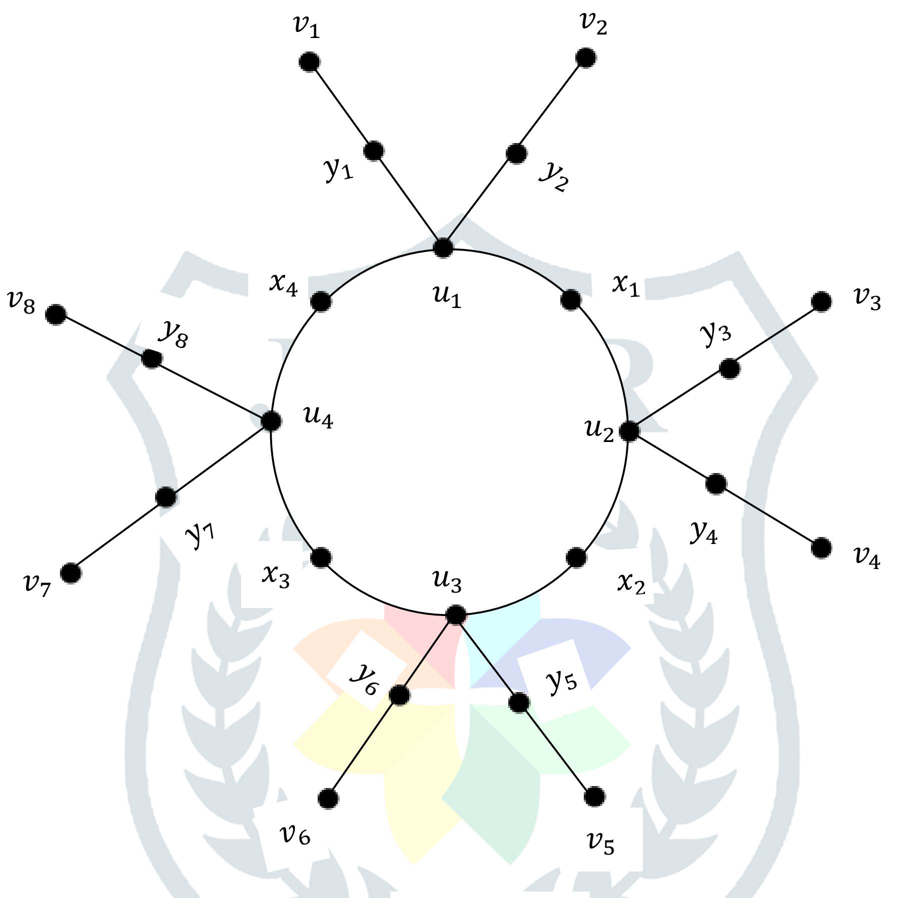

figure : 5

Theorem 3.8

Independent domination number of the subdivision graph $S(DW_{2n+1})$ if n ≥ 3 then

$$i[S(DW_{2n+1})] = \begin{cases} \left\lceil \frac{4n+3}{3} \right\rceil, & if\ n \cong 0 (mod\ 3) and\ n \cong 1\ (mod\ 3) \\ \left\lceil \frac{4n+6}{3} \right\rceil, & if\ n \cong 2 (mod\ 3) \end{cases}$$

Proof:

The subdivision graph of Double Wheel graph DW_{2n+1} contains the subgraphs outer cycle $S(C_n)$, the inner cycle $S(C_n)$ and the hub of the wheel respectively.

The graph $S(DW_{2n+1})$ contains $6n + 1$ vertices. Among which $2n$ vertices are actual vertices, one vertex is the hub of wheel and the remaining vertices are the subdivision vertices.

Therefore, for all the values of $n, n\ n \geq 3$.

The $2n$ vertices are of degree 3. The hub of the wheel is of degree 2n. The subdivision vertices are of degree 2.

Case (1) $n \cong 0 (mod\ 3) and\ n \cong 1\ (mod\ 3)$

Let u_i, $1 \leq i \leq n$ be the vertices of inner most cycle of DW_{2n+1} and v_i, $1 \leq i \leq n$ be the vertices of outer most cycle of DW_{2n+1}. Let x_i, $1 \leq i \leq n$ be the subdivision vertices of inner cycle of DW_{2n+1} and y_i, $1 \leq i \leq n$ be the subdivision vertices of outer most cycle of DW_{2n+1}.

Let c denote the hub of the wheel. Let c_i, $1 \leq i \leq n$ be the subdivision vertices in the spokes of the inner wheel and c_i^1, $1 \leq i \leq n$ be the subdivision vertices in the spokes of the outer wheel.

Vertex u_i, $2 \leq i \leq n$ is adjacent to x_{i-1} and x_{i+1}, c_i whereas c_i is adjacent to the hub of the wheel and v_i, $2 \leq i \leq n$ is adjacent to y_{i-1} and y_{i+1}, c_i^1 whereas c_i^1, $1 \leq i \leq n$ is adjacent to the hub of the wheel.

Since deg (c) = $\Delta(DW_{2n+1})$ = 2n and the vertex c dominate 2n subdivision vertices in the spokes of the inner and outer wheel of S(DW_{2n+1}). Therefore, for any independent dominating set should contain the vertex c.

The graph S(DW_{2n+1}) contains two cycle with 2n vertices. Now by Result 2.9, i(C_{2n})= $\lceil 2n/3 \rceil$. Therefore, at least 2 $\lceil 2n/3 \rceil$ vertices are required to dominate all the vertices of inner and outer cycle of S(DW_{2n+1}).

Hence, at least 2$\lceil 2n/3 \rceil$ + 1 = $\left\lceil \frac{4n+3}{3} \right\rceil$ vertices are essential to dominate all the vertices of S(DW_{2n+1}). Therefore, for any independent dominating set S of S(DW_{2n+1}), $|S| \geq \left\lceil \frac{4n+3}{3} \right\rceil$ which implies that i[S(DW_{2n+1})] = $\left\lceil \frac{4n+3}{3} \right\rceil$.

Case (2) $n \cong 2(mod\ 3)$

Let V(S(DW_{2n+1})) = {c, u_i,v_i, x_i, y_i, c_i, c_i^1}.

In the same process as in case (1). Therefore, at least 2$\lceil 2n/3 \rceil$ + 1 = $\left\lceil \frac{4n+6}{3} \right\rceil$ vertices are enough to dominate all the vertices of S(DW_{2n+1}).

Hence, for any independent dominating set S of S(DW_{2n+1}), $|S| \geq \left\lceil \frac{4n+6}{3} \right\rceil$ which implies that i[S(DW_{2n+1})] = $\left\lceil \frac{4n+6}{3} \right\rceil$.

Example 3.9

In figure 6, the graph obtained by subdivision of each edge of DW_{11} in which the set of solid vertices is its minimum independent dominating set. Therefore, i[S(DW_{11})] = 9.

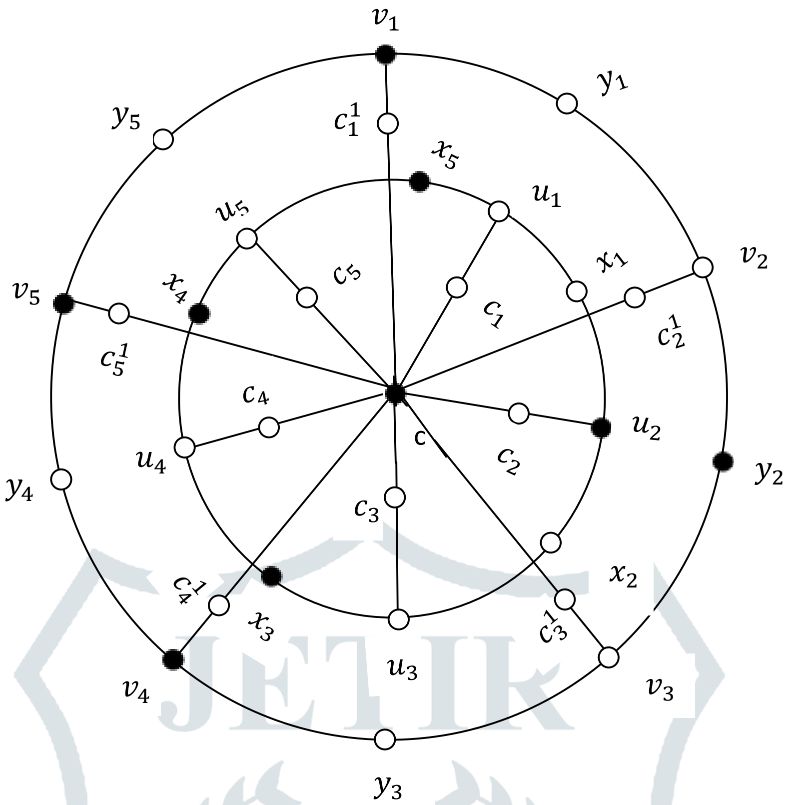

figure : 6

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