Perfect Domination Polynomial under Graph operation

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Abstract : **Abstract:** Let $G=(V,E)$ be a connected simple undirected graph. A subset S⊆ V is a dominating set if for every vertex $v \in V/S$, there exists a vertex $w \in S$ such that $vw \in E$; that is every vertex outside S has atleast one neighbor in S. The minimum cardinality of a dominating set of G is the domination number of G, it is denoted by $\gamma(G)$. A subset S⊆ V is a perfect dominating set if for every vertex $v \in V/S$, there exists exactly one vertex $w \in S$ such that $vw \in E$; that is every vertex outside S has exactly one neighbor in S. The minimum cardinality of a perfect dominating set of G is the perfect domination number of G, it is denoted by $\gamma_p(G)$. We introduce new concept that a perfect domination polynomial of a graph G. The perfect domination polynomial of a graph G of order n is the polynomial

$$PD(G,x)=\sum_{i=\gamma_p(G)}^{n} pd(G,i)x^i$$

Where $pd(G,i)$ is the number of perfect dominating sets of G of size i and $\gamma_p(G)$ is the perfect domination number of G. we obtain some properties of D(G,x) for component graph and then derive some concepts based on it.

***IndexTerms* - Domination set, Domination polynomial, Lollipop graph, Perfect domination, Perfect domination polynomial.**

I. Introduction

In literature, the concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [6] is currently receiving much attention. Following the article of Ernie Cockayne and Stephen Hedetniemi [2], the domination in graphs became an area of study by many researchers. One type of domination in graphs is the perfect domination. This was introduced by Cockayne et al. [1] in the paper Perfect domination in graphs.

Let $G=(V,E)$ be a connected simple graph and $v \in V$. The open neighbourhood of v is the set $N(v)=\{u \in V; uv \in E\}$. The closed neighbourhood of v is the set $N[v]=N(v)\cup\{v\}$. A subset S⊆ V is a dominating set if for every vertex $v \in V/S$, there exists a vertex $w \in S$ such that $vw \in E$; that is every vertex outside S has atleast one neighbor in S. The minimum cardinality of a dominating set of G is the domination number of G, it is denoted by $\gamma(G)$. A subset S⊆ V is a perfect dominating set if for every vertex $v \in V/S$, there exists exactly one vertex $w \in S$ such that $vw \in E$; that is every vertex outside S has exactly one neighbor in S. The minimum cardinality of a perfect dominating set of G is the perfect domination number of G, it is denoted by $\gamma_p(G)$. Let D(G, i) be the family of dominating sets of a graph G with cardinality i and let d(G, i) = |D(G, i)|. Then the domination polynomial D(G, x) of G is defined as $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$

In the second section, we introduce the domination polynomial and in the third section we derive perfect domination polynomial for the Lollipop graph.

I. PERFECT DOMINATION POLYNOMIAL

The perfect domination polynomial of a graph G of order n is the polynomial

$$PD(G,x)=\sum_{i=\gamma_p(G)}^{n} pd(G,i)x^i$$

Where $pd(G,i)$ is the number of perfect dominating sets of G of size i and $\gamma_p(G)$ is the perfect domination number of G.

Theorem 1.1:

If G is a Graph without isolated vertices, consisting of two components G_1 and G_2, then PD (G, x) = PD (G_1, x). PD(G_2, x).

Proof:

Let G_1 and G_2 be the components of a Graph G without isolated vertices. Let the Vertex-edge domination number of G_1 and G_2 be $\gamma_p(G_1)$ and $\gamma_p(G_2)$. For any $k \geq \gamma_p(G)$, the vertex-edge dominating set of k vertices in G arises by choosing a vertex-edge dominating set of j vertices of G_1 and a Vertex-edge dominating set of k-j vertices in G_2.

The number of vertex-edge dominating sets in $G_1 \cup G_2$ is equal to the co-efficient of x^k in $PD(G_1, x). PD(G_2, x)$. But the number of vertex-edge dominating sets of G of cardinality k is the co-efficient of x^k in $PD(G,x)$.

Hence the co-efficient of x^k in $PD(G, x)$ and $PD(G_1, x). D(G_2, x)$ are equal.

$$\therefore PD(G, x) = PD(G_1, x). PD(G_2, x).$$

Theorem 1.2

If G is a Graph without isolated vertices, consisting of m components G_1 and G_2, then

$$PD(G, x) = PD(G_1, x). PD(G_2, x) \ldots\ldots PD(G_m, x)$$

Proof:

The proof of the theorem follows from theorem 1.1.

Theorem 1.3:

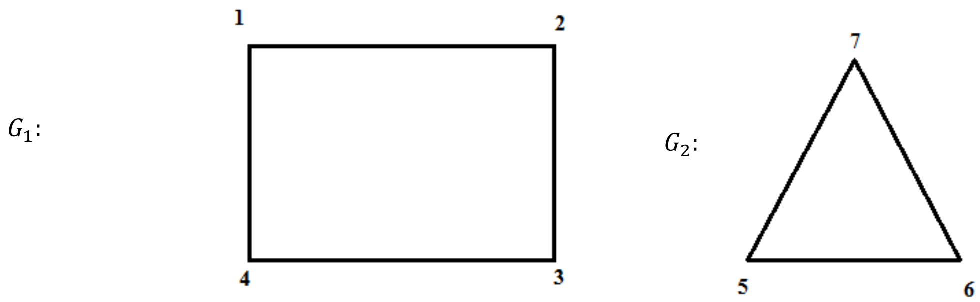
If G_1 and G_2 are Graphs without isolated vertices, let $G = G_1 \cup G_2$, then $PD(G, x) = PD(G_1, x). PD(G_2, x)$.

Proof:

The proof of the theorem follows from theorem 1.1.

Example:

Consider the graph given in the figure 1:



$G = G_1 \cup G_2$

Figure 1

Consider G_1:

Here Perfect domination number of G_1 is 2. (i.e., $\gamma_p(G_1) = 2$). So $pd(G_1, 1) = 0$.
Perfect domination set of size 2 are $\{1,4\}, \{2,3\}$. So $pd(G_1, 2) = 2$.
There is no Perfect domination set of size 3, so $pd(G_1, 3) = 0$.
Perfect domination set of size 4 are $\{1,2,3,4\}$. So $pd(G_1, 4) = 1$.

Therefore the perfect domination polynomial is defined by

$$PD(G_1, x) = 0x^1 + 2x^2 + 0x^3 + 1x^4 = 2x^2 + 1x^4 = x^2(2 + x^2)$$

Consider G_2:

Here Perfect domination number of G_2 is 1. (i.e., $\gamma_p(G_2) = 1$). .
Perfect domination set of size 1 are $\{5\}, \{6\}, \{7\}$. So $pd(G_2, 1) = 3$.
There is no Perfect domination set of size 2, so $pd(G_2, 2) = 0$.

Perfect domination set of size 3 are {5,6,7}. So $pd(G_2,3) = 1$.

Therefore the perfect domination polynomial is defined by

$$PD(G_2,x) = 3x^1 + 0x^2 + 1x^3 = 3x^1 + 1x^3 = x^1(3 + x^2)$$

Consider $G = G_1 \cup G_2$:

Here Perfect domination number of G is 3. (i.e., $\gamma_p(G) = 3$). So $pd(G,1) = pd(G,2) = pd(G,2) = 0$.
Perfect domination set of size 3 are {2,4,5}, {2,4,6}, {2,4,7}, {1,3,5}{1,3,6}, {1,3,7} .So $pd(G,3) = 6$.
There is no Perfect domination set of size 4, so $pd(G,4) = 0$.
Perfect domination set of size 5 are {1,3,5,6,7}, {2,4,5,6,7}{1,2,3,4,5}{1,2,3,4,6}, {1,2,3,4,7}.
So $pd(G,5) = 5$
There is no Perfect domination set of size 6, so $pd(G,6) = 0$
Perfect domination set of size 7 are {1,2,3,4,5,6,7}, So $pd(G,7) = 1$

Therefore the perfect domination polynomial is defined by

$$\begin{aligned} PD(G_2,x) &= 0x^1 + 0x^2 + 6x^3 + 0x^4 + 5x^5 + 0x^6 + x^7 \\ &= 6x^3 + 5x^5 + x^7 \\ &= x^3(6 + 5x^2 + x^4) \\ &= x^3(3 + x^2)(2 + x^2) \\ &= PD(G_1,x)PD(G_2,x) \end{aligned}$$

By proceeding in this manner we get the general term for m components,

PD (G, x) = PD (G_1, x). PD(G_2, x)…… PD(G_m, x)

Theorem 1.4:

Let G be a graph with $|V(G)| = n$. If G is connected, then in the perfect domination polynomial

i. the coefficient of x^n is 1 and

ii. the coefficient of x^{n-1} is n.

Proof:

(i) Since G has n vertices, there is only one way to choose all these vertices and it dominates all the vertices and edges. Therefore, $pd(G,n) = 1$.

(ii) If we delete one vertex v, the remaining n – 1 vertices dominate all the vertices and edges of G. (This is done in n ways). Therefore, pd (G, n – 1) = n.

Theorem 1.5:

Let G be a graph with $|V(G)| = n$. If G is connected, then in the perfect domination polynomial $pd(G,i) = 0$ if $i < \gamma_{pd}(G)$.

Proof:

(i) Since $D_{ve}(G, i) = \phi$ if $i < \gamma_{ve}(G)$

Therefore, we have $d_{ve}(G, i). = 0$ if $i < \gamma_{ve}(G)$

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