FIXED POINT THEOREM FOR HYBRID CONTRACTIONS

Ritu Arora

ABSTRACT

The purpose of this paper is to obtain coincidence and fixed point theorems for a pair of hybrid contracting maps.

Key words and phrases: Coincidence point, fixed point, hybrid contracting maps, hybrid contraction.

1. INTRODUCTION

Consistent with [1], [4], [11], [16], [18], [20] and [23], we will use the following notations, where (X, d) is a metric space. Let $CL(X)$ denote the collection of all nonempty closed subsets of X. The distance function H on $CL(X)$ is called the generalized Hausdorff metric induced by the metric d of X. Further, let $d(A, B)$ denote the ordinary distance between nonempty subsets A and B of X, while $d(A, x)$ stands for $d(A, B)$ when B is the singleton $\{x\}$.

Let $T : X \to CL(X)$ be a multivalued map and $f : X \to X$ a single-valued map. Consider the following condition essentially due to Singh and Kulshrestha [21].

$$H(Tx, Ty) \leq q.\max\{d(fx, fy), d(fx, Tx), (fy, Ty), [d(fx, Ty) + d(fy, Tx)]/2\} \qquad \text{(SK)}$$

for all $x, y \in X$, where $q \in (0, 1)$.

We remark that (SK) with f = the identity map on X is Ćirić's generalized multivalued contraction (Ćirić [4]), which in turn includes the well-known Nadler multivalued contraction (Nadler, Jr. [16], see also [1], [4], [9]-[13], [15], [23] and [25]).

The study of hybrid contracting maps involving single-valued and multivalued maps on metric spaces was initiated independently by Bhaskaran and Subrahmanyam [2], Hadžić [5], Kaneko [9], Kubiak [15], Naimpally [17] *et al*. and Singh and Kulshrestha [21]. (Here, according to Singh and Mishra [22], "hybrid contracting maps" means a pair of hybrid contraction or nonexpansive or contractive). Indeed, Singh and Kulshrestha [*op. cit.*] (see also [12], [19] and [22]) showed that T and f satisfying $T(X) \subseteq f(X)$ and (SK) have a coincidence, that is, there exists a point $z \in X$ such that $fz \in Tz$ when $f(X)$ is a complete subspace of X. This result is obviously true when, instead of $f(X)$, $T(X)$ is a complete subspace of X . For an immediate excellent generalization of this result one may refer to Rhoades *et al*. [18].

In all that follows, Y is an arbitrary nonempty set and (X, d) a metric space. Following Liu *et al*. [14], Singh and Mishra [23] and Tan *et al*. [26], we shall consider the following condition for $f : Y \to X$ and
$T : Y \to CL(X)$.

$$H^2(Tx, Ty) \leq q.m(x, y), \tag{1.1}$$

where $q \in (0, 1)$ and

$$\begin{aligned} m(x, y) := \max \{ & d^2(fx, fy), d(fx, fy).d(fx, Tx), d(fx, fy).d(fy, Ty), \\ & d(fx, fy).[d(fx, Ty) + d(fy, Tx)]/2, d(fx, Tx).d(fy, Ty), \\ & d(fx, Tx).[d(fx, Ty) + d(fy, Tx)]/2, d(fy, Ty).[d(fx, Ty) \\ & + d(fy, Tx)]/2, d(fx, Ty).d(fy, Tx)\}. \end{aligned}$$

Our main result is under the condition (1.1) (cf. Theorem 3). Following Rhoades *et al*. [18], we present generalized versions of this result as well.

2. RESULTS

DEFINITION 1. (Itoh and Takahashi [6], see also Singh and Mishra [24, p. 488]). Let Y be a nonempty set, $f : Y \to Y$ and $T : Y \to 2^Y$, the collection of all nonempty subsets of Y. Then the hybrid pair (T, f) is IT- commuting at $x \in Y$ if $fTx \subseteq Tfx$ for each $x \in Y$. (This formulation in [25] is correct with the interchange of symbols for single-valued and mutivalued maps).

We shall need the following result, which is a minor variant of a lemma due to Ćirić [4].

LEMMA 2. Let $A, B \in CL(X)$. Then for an $x \in A$ and for some q and k in (0, 1), there exists a $y \in B$ such that

$$d^2(x, y) \le q^{-k} H^2(A, B).$$

THEOREM 3. Let Y be an arbitrary nonempty set and (X, d) a metric space. Let $T : Y \to CL(X)$ and $f : Y \to X$ be such that $T(Y) \subseteq f(Y)$ and (1.1) holds for all $x, y \in Y$. If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that

(I) $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$;

(II) $\{fx_n\}$ converges to fz for some $z \in Y$, and $fz \in Tz$,
that is, T and f have a coincidence at z; and

(III) $d(fx_n, fz) \le [\beta^n/(1-\beta)].d(fx_0, fx_1)$, where $\beta = q^{1-k}[1 + \sqrt{(1 + 8q^{-1+k})}]/4$
for some $k \in (0, 1)$.

Further, if $Y = X$ and the pair (T, f) is IT-commuting at z then T and f have a common fixed point provided that $ffz = fz$.

PROOF. Pick a point x_0 in Y. Let k be a positive number such that $k < 1$. Following Singh and Kulshrestha [21] (see also [12]), we construct sequences $\{x_n\} \subseteq Y$ and $\{fx_n\} \subseteq X$ in the following manner.

Since $T(Y) \subseteq f(Y)$, we may choose a point $x_1 \in Y$ such that
$fx_1 \in Tx_0$. If $Tx_0 = Tx_1$ then $x_1 = z$ is a coincidence point of T and f, and we are done. So assume that $Tx_0 \ne Tx_1$ and choose $x_2 \in Y$ such that
$fx_2 \in Tx_1$ and, by Lemma 2,

$$d^2(fx_1, fx_2) \le q^{-k} H^2(Tx_0, Tx_1).$$

If $Tx_1 = Tx_2$, then x_2 becomes a coincidence point of T and f. If not, continue the process. In general, if $Tx_n \neq Tx_{n+1}$, we choose $fx_{n+2} \in Tx_{n+1}$ such that

$$d^2(fx_{n+1}, fx_{n+2}) \leq q^{-k} H^2(Tx_n, Tx_{n+1}).$$

Then by (1.1),

$d^2(fx_{n+1}, fx_{n+2}) \leq q^{-k}H^2(Tx_n, Tx_{n+1})$

$\leq q^{1-k}\max \{d^2(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}).d(fx_n, Tx_n), d(fx_n, fx_{n+1}).d(fx_{n+1}, Tx_{n+1}),$

$d(fx_n, fx_{n+1}).[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)]/2, d(fx_n, Tx_n).d(fx_{n+1}, Tx_{n+1}),$

$d(fx_n, Tx_n).[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)]/2, d(fx_{n+1}, Tx_{n+1}).[d(fx_n, Tx_{n+1})$

$+ d(fx_{n+1}, Tx_n)]/2, d(fx_n, Tx_{n+1}. d(fx_{n+1}, Tx_n)\}.$

For the sake of simplicity, let $d_n := d(fx_n, fx_{n+1})$.

Then the above inequality, after simplification, reduces to

$$d_{n+1}^2 \leq q^{1-k}. \max \{d_n^2, d_nd_{n+1}, d_n[d_n + d_{n+1}]/2, d_{n+1}[d_n + d_{n+1}]/2\}. \qquad (3.1)$$

We remark that in the construction of sequences $\{x_n\}$ and $\{fx_n\}$, x_n (for each n) is not a coincidence point of T and f. This together with

$Tx_n \neq Tx_{n+1}$ means that $fx_n \neq fx_{n+1}$. Indeed, if at any stage $fx_n = fx_{n+1}$ then

$fx_n \in Tx_n$, and x_n is a coincidence point of T and f. Therefore, according to our construction of the sequences, $d_n \neq 0$. Hence the inequality (3.1) implies one of the following:

$d_{n+1} \leq \lambda\, d_{n+1}$, where $\lambda = q^{1-k}$;

$d_{n+1}^2 \leq \lambda\, d_n^2$;

$d_{n+1} \leq [\lambda/4 + \sqrt{}(\lambda^2/16 + \lambda/2)]d_n$;

$d_{n+1} \leq [\lambda/(2 - \lambda)]d_n$.

Consequently,

$$d_{n+1} \leq \max \{\sqrt{}\lambda, \lambda, [\lambda/4 + \sqrt{}(\lambda^2/16 + \lambda/2)], [\lambda/(2 - \lambda)]\}\, d_n.$$

This gives $d_{n+1} \leq \beta d_n$, when $\beta = [\lambda/4 + \sqrt{}(\lambda^2/16 + \lambda/2)]$. Notice that

$0 < \beta < 1$. Hence $\{fx_n\}$ is a Cauchy sequence.

Now let $f(Y)$ be a complete subspace of Y. Then the sequence $\{fx_n\}$ has a limit $f(Y)$. Call it b. Hence there exists a point $z \in Y$ such that $fz = b$.

Since $d(fz, Tz) \leq d(fz, fx_{n+1}) + H(Tx_n, Tz)$, applying (1.1) to the last term of this inequality and making $n \to \infty$, the inequality gives

$d(fz, Tz) \leq \sqrt{q}.d(fz, Tz)$, and $fz \in Tz$, since $\sqrt{q} < 1$ and Tz is closed. This argument applies to the case when $T(Y)$ is a complete subspace of Y, since $T(Y) \subseteq f(Y)$.

This proves (I) and (II). To see (III), let $m > n$. Then,

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n+1}) + \ldots + d(fx_{m+1}, fx_m)$$

$$\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1}).\, d(fx_0, fx_1)$$

$$< \beta^n/(1 - \beta).d(fx_0, fx_1).$$

Making $m \to \infty$, we get (III).

REFERENCES

1. Ismat Beg and Akbar Azam, Fixed points of asymptotically regular multivalued mappings, Internat, J. Math. Math. Sci. 15 (1992), 15-30.

2. R. Bhaskaran and P. V. Subrahmanyam, Common fixed points in metrically convex spaces, J. Math. Phys. Sci. 18 (5) (1984), 65-70.

3. H. W. Corley, Some hybrid fixed point theorems related to optimization, J. Math. Anal. Appl. 120 (1986), 528-532.

4. Lj. B. Ćirić, Fixed points for generalized multivalued contractions, Math. Vesnik, 9 (24) (1972), 265-272.

5. O. Hadžić, A coincidence theorem for multivalued mappings in metric spaces, Studia Univ. Babes-Bolyai Math. 26(1981), No. 4, 65-67.

6. S. Itoh and W. Takahashi, Single-valued mappings, multivalued mappings and fixed point theorems, J. Math. Anal. Appl. 59 (1977), 514-521.

7. J. Jachymski, On Reich's question concerning fixed points of multimap, Boll. Un. Mat. Ital. (7) 9-A (1995), 453-460.

8. G. Jungck and B. E. Rhoadeds, Fixed points for set-valued functions without continuity, Indian J. pure Appl. Math. 29 (1998), 227-238.

9. H. Kaneko, Single valued and multivalued f-contractions, Boll. Un. Mat. Ital. (6) 4-A (1985), 29-33.

10. H. Kaneko, A comparison of contractive conditions for multivalued mappings, Kobe J. Math. 3 (1986), 37- 45.

11. H. Kaneko and S. Seesa, Fixed point theorems for compatible multivalued and single valued mappings, Internat. J. Math. Sci. 12 (1989), 257-262.

12. C. Kulshrestha, Single-valued mappings, multi-valued mappings and fixed points theorems in

metric spaces, Ph.D. Thesis (supervised by Prof. S. L. Singh), Garhwal University, Srinagar, 1983.

13. T. Kubiak, Two coincidence theorems for contractive type multivalued mappings, Studia Univ. Babes-Bolyai Math. 30 (1985), 65-68.

14. Z. Liu, F. Zhang and J. Mao, Common fixed points for compatible mappings of type (A), Bull. Malaysian Math. Soc. 22 (1999), 67-86.

15. S. N. Mishra, S. L. Singh and Rekha Talwar, Nonlinear hybrid contractions on Menger and Uniform spaces, Indian J. Pure Appl. Math. 25 (1994), 1039-1052.

16. S. B. Nadler, Jr., Multivalued contraction mappings, Pacific J. Math.30 (1969), 475-488.

17. S. A. Naimpally, S. L. Singh and J. H. M. Whitfield : Coincidence theorems for hybrid contractions, Math. Nachr. 127(1986), 177-180.

18. B. E. Rhoades, S. L. Singh and C. Kulshrestha, Coincidence theorems for some multivalued mappings, Internat. J. Math. Math. Sci. 7(1984), 429-434.

19. K. P. R Sastry, I. H. N. Rao and K. P. R. Rao, A fixed point theorem for mutimaps, Indian J. Phy. Natur. Sci. 3 (sec B), (1983), 1-4.

20. S. L. Singh, K. S. Ha and Y. J. Cho, Coincidence and fixed points of nonlinear hybrid contractions, Internat. J. Math. Math. Sci. 12 (1989), 247-256.

21. S. L. Singh and Kulshrestha, Coincidence theorem in metric spaces, Indian J. Phy. Natur. Sci. 2B (1982), 19-22.

22. S. L. Singh and S. N. Mishra, Nonlinear hybrid contractions, Journal of Natural & Physical Sciences 5-8 (1993), 191-206.

23. S. L. Singh and S. N. Mishra, On general hybrid contractions, J. Austral. Math. Soc. (Series A) 66 (1999), 244-254.

24. S. L. Singh and S. N. Mishra, Coincidences and fixed points of nonself hybrid contractions, J. Math. Anal. and Appl. 256 (2001), 486-497.

25. S. L. Singh and S. N. Mishra, Coincidences and fixed points of reciprocally continuous and compatible hybrid maps, Internat J. Math. Sci. 30 (2002), 627-635.

26. Dejun Tan, Zeqing Liu and Jong Kyu Kin, Common fixed points for compatible mappings of Type (P) in 2-metric spaces, Nonlinear Funct. Anal. & Appl., Vol. 8, NO. 2 (2003), 215-232.