Statistical distribution and Pathway integral Operator of p-k Mittag-Leffler Function

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Abstract

The aim of this paper is to investigate the pathway integral operator of the p-k Mittag-Leffler function. We also investigate statistical distribution associated with the p-k Mittag-Leffler function and its Order and type. Certain particular cases of the derived results are considered and indicated to further reduce to some known results.

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1 Introduction

The two parameter pochhammer symbol is recently introduce by [5], equation 2.1, in the form,

1.1 Definition

Let $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by,

$${}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p).........(\frac{xp}{k} + (n-1)p). \tag{1}$$

And the Two Parameter Gamma Function is given by [5], equation 2.6, 2.7 and 2.14.

1.2 Definition

For $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k(x)$ as,

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \tag{2}$$

or,

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1} (np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \tag{3}$$

The integral representation of p - k Gamma Function is given by,

$${}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \tag{4}$$

The p - k Mittag-Leffler function [4], denoted by ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$$ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{z^n}{n!}. \quad (5)$$

Where $k,p\in R^+-\{0\};\alpha,\beta,\gamma\in C/kZ^-;Re(\alpha)>0,Re(\beta)>0,Re(\gamma)>0$ and $q\in(0,1)\cup N$.

${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1.1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by equation (1.3).

Let $f\in L(a,b),\eta\in C\ with\ R(\eta),a\in R^+,and\ s<1$ be the pathway parameter. Then the pathway fractional integration operator is defined by Nair[11] as,

$$\left(P_{0+}^{(\eta,s)}f\right)(x)=x^{\eta}\int_0^{\left[\frac{x}{a(1-s)}\right]}[1-\frac{a(1-s)t}{x}]^{\frac{\eta}{1-s}}f(t)dt. \quad (6)$$

For a real scalar s, the pathway model for scalar random variables is represented by the following probability density function:

$$f(x)=c\,|x|^{\nu-1}[1-a(1-s)|x|^{\xi}]^{\frac{\lambda}{1-s}}.$$

Provided that $x\in R,\quad \nu,\xi\in R^+,\lambda\in R_0^+,1-a(1-s)|x|^{\xi}>0$. Here c is the normalizing constant, and s is called the pathway parameter.

For $s>1$, (6) can be written as follows:

$$\left(P_{0+}^{(\eta,s)}f\right)(x)=x^{\eta}\int_0^{\left[\frac{x}{-a(1-s)}\right]}\left[1+\frac{a(s-1)t}{x}\right]^{\frac{\eta}{-(s-1)}}f(t)dt. \quad (7)$$

And,

$$f(x)=c\,|x|^{\nu-1}[1+a(s-1)|x|^{\xi}]^{-\frac{\lambda}{(s-1)}}. \quad (8)$$

Provided that $x\in R,\quad \nu,\xi\in R^+,\lambda\in R_0^+$.

Moreover,as $s\to1$, the operator (6) reduces to the Laplace integral transform, and when $s=0$ and a=1, replacing $\eta\ by\ \eta-1$, the operator (6) reduces to the Riemann-Liouville fractional integral operator.

2 p-k Mittag-Leffler function and statistical distribution

We investigate the density function for p-k Mittag-Leffler function.

Theorem 1 let $k,p,q,\mu\in R^+-\{0\}$ with $0<\mu\le1$ and $x\in C/kZ^-$. Also, let $\gamma\in C$,

$$F_x(x)=1-{}_pE^{\gamma,q}_{k,\mu,k}(-x^{\mu})$$

. Then the density function f(x) of $F_x(x)$ is givan as follows:

$$f(x)=\mu\quad x^{\mu-1}\quad {}_p(\gamma)_{q,k}\quad {}_pE^{\gamma+qk,q}_{k,\mu,\mu+k}(-x^{\mu}). \quad (9)$$

Proof Using (5), we have,

$$F_x(x)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(\mu n+k)}\frac{(-x^{\mu})^n}{n!}$$

. Differentiating with respect to x gives the density function,

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\quad {}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(\mu n+k)}\frac{\mu n\quad x^{\mu n-1}}{n!}$$

. Replace n by n+1, yields,

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n \quad {}_p(\gamma)_{nq+q,k}}{{}_p\Gamma_k(\mu n+\mu+k)}\frac{\mu n+\mu \quad x^{\mu n+\mu-1}}{n+1!}$$

. Applying the relation,

$${}_p(x)_{n+j,k}={}_p(x)_{j,k}\times{}_p(x+jk)_{n,k}$$

. We get,

$$f(x)=x^{\mu-1} \quad \mu \quad \sum_{n=0}^{\infty}\frac{(-1)^n \quad {}_p(\gamma)_{q,k} \quad {}_p(\gamma+qk)_{nq,k}}{{}_p\Gamma_k(\mu n+\mu+k)}\frac{x^{\mu n}}{n!}$$

,

$$f(x)=\mu \quad x^{\mu-1} \quad {}_p(\gamma)_{q,k} \quad {}_pE_{k,\mu,\mu+k}^{\gamma+qk,q}(-x^{\mu})$$

.

3 Special Cases

We consider some particular cases of theorem 1.

Corollary 3.1 If we substitued $p=k$ in equation (9), then,

$$f(x)=\mu \quad x^{\mu-1} \quad {}_k(\gamma)_{q,k} \quad {}_kE_{k,\mu,\mu+k}^{\gamma+qk,q}(-x^{\mu})$$

,

$$f(x)=\mu \quad x^{\mu-1} \quad (\gamma)_{q,k} \quad E_{k,\mu,\mu+k}^{\gamma+qk,q}(-x^{\mu}). \tag{10}$$

Which is new result.

Corollary 3.2 If we substitued $p=k$ and $q=1$ in equation (9), then,

$$f(x)=\mu \quad x^{\mu-1} \quad {}_k(\gamma)_{1,k} \quad {}_kE_{k,\mu,\mu+k}^{\gamma+k,1}(-x^{\mu})$$

,

$$f(x)=\mu \quad x^{\mu-1} \quad (\gamma)_{1,k} \quad E_{k,\mu,\mu+k}^{\gamma+k}(-x^{\mu}). \tag{11}$$

This is known result given by [1] (equation (1.9)).

Corollary 3.3 If we substitued $p=k=1$ and $q=1$ in equation (9), then,

$$f(x)=\mu \quad x^{\mu-1} \quad {}_1(\gamma)_{1,1} \quad {}_1E_{1,\mu,\mu+1}^{\gamma+1,1}(-x^{\mu})$$

,

$$f(x)=\mu \quad x^{\mu-1} \quad (\gamma) \quad E_{\mu,\mu+1}^{\gamma+1}(-x^{\mu}). \tag{12}$$

This is known result given by [1] (equation (1.11)).

4 Pathway integral reprensentation of p-k Mittag-Leffler Function

Theorem 2 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{1-\sigma}>-1$ Also, let $k,\sigma\in R$ with $\sigma<1$ and $p,q\in R^{+}$. then,

$$P_{0+}^{\eta,\sigma}[t^{\frac{\beta}{k}-1} \quad {}_pE_{k,\alpha,\beta}^{\gamma,q}(\omega t^{\frac{\alpha}{k}})(x)]$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{(a(1-\sigma))^{\frac{\beta}{k}}}P^{\frac{\eta}{1-\sigma}+1} \quad \Gamma(\frac{\eta}{1-\sigma}+1)\,{}_pE_{k,\alpha,\frac{\eta}{1-\sigma}k+k+\beta}^{\gamma,q}[\omega(\frac{x}{a(1-\sigma)})^{\frac{\alpha}{k}}]. \tag{13}$$

Proof Using equation (5) and (6), we have,

$$P_{0+}^{\eta,\sigma}[t^{\frac{\beta}{k}-1}\,{}_pE_{k,\alpha,\beta}^{\gamma,q}(\omega t^{\frac{\alpha}{k}})(x)]$$

$$= x^{\eta}\int_0^{\frac{x}{a(1-\sigma)}}(1-\frac{a(1-\sigma)t}{x})^{\frac{\eta}{1-\sigma}}t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{(\omega t^{\frac{\alpha}{k}})^n}{n!}dt.$$

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{\omega^n}{n!}\int_0^{\frac{x}{a(1-\sigma)}}(1-\frac{a(1-\sigma)t}{x})^{\frac{\eta}{1-\sigma}}t^{\frac{n\alpha}{k}+\frac{\beta}{k}-1}dt.$$

Put $\frac{a(1-\sigma t)}{x}=v$ then,

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{\omega^n}{n!}\int_0^{1}(1-v)^{\frac{\eta}{1-\sigma}}(\frac{x}{a(1-\sigma)})^{\frac{n\alpha}{k}+\frac{\beta}{k}-1}v^{\frac{n\alpha}{k}+\frac{\beta}{k}-1}\frac{x}{a(1-\sigma)}dv.$$

Using definition of beta function,

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{\omega^n}{n!}(\frac{x}{a(1-\sigma)})^{\frac{n\alpha}{k}+\frac{\beta}{k}}\frac{\Gamma(\frac{\eta}{1-\sigma}+1)\Gamma(\frac{n\alpha}{k}+\frac{\beta}{k})}{\Gamma(\frac{\eta}{1-\sigma}+\frac{n\alpha}{k}+\frac{\beta}{k})}$$

. Using the relation between p-k Gamma Function and classical Gamma Function [5] (equation 2.19), we have,

$$=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{1-\sigma}+1)}{(a(1-\sigma))^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}\,p^{\frac{\eta}{1-\sigma}+1}}{{}_p\Gamma_k((\frac{\eta}{1-\sigma})k+k+\beta+n\alpha)}(\omega(\frac{x}{a(1-\sigma)})^{\frac{\alpha}{k}})$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{(a(1-\sigma))^{\frac{\beta}{k}}}P^{\frac{\eta}{1-\sigma}+1}\;\Gamma(\frac{\eta}{1-\sigma}+1)\,{}_pE_{k,\alpha,\frac{\eta}{1-\sigma}k+k+\beta}^{\gamma,q}[\omega(\frac{x}{a(1-\sigma)})^{\frac{\alpha}{k}}].$$

This establish result(13).

Corollary 4.1 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{1-\sigma}>-1$ Also, let $\sigma\in R$ with $\sigma<1$, $p,q\in R^{+}$ and $p=k$ then theorem 2 reduces to,

$$P_{0+}^{\eta,\sigma}[t^{\frac{\beta}{k}-1}\,{}_kE_{k,\alpha,\beta}^{\gamma,q}(\omega t^{\frac{\alpha}{k}})(x)]$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{(a(1-\sigma))^{\frac{\beta}{k}}}k^{\frac{\eta}{1-\sigma}+1}\;\Gamma(\frac{\eta}{1-\sigma}+1)\,{}_kE_{k,\alpha,\frac{\eta}{1-\sigma}k+k+\beta}^{\gamma,q}[\omega(\frac{x}{a(1-\sigma)})^{\frac{\alpha}{k}}]. \qquad (14)$$

Which is new result.

Corollary 4.2 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{1-\sigma}>-1$ Also, let $\sigma\in R$ with $\sigma<1$, $p,q\in R^{+}$ and $p=k=1$ then theorem 2 reduces to,

$$P_{0+}^{\eta,\sigma}[t^{\beta-1}\,{}_1E_{1,\alpha,\beta}^{\gamma,q}(\omega t^{\frac{\alpha}{1}})(x)]$$

$$=\frac{x^{\eta+\beta}}{(a(1-\sigma))^{\beta}}\quad\Gamma(\frac{\eta}{1-\sigma}+1)_1E^{\gamma,q}_{1,\alpha,\frac{\eta}{1-\sigma}+1+\beta}[\omega(\frac{x}{a(1-\sigma)})^{\alpha}]. \qquad (15)$$

Which is new result.

Corollary 4.3 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{1-\sigma}>-1$ Also, let $\sigma\in R$ with $\sigma<1$, $p,q\in R^{+}$, $p=k=1$, and $q=1$ then theorem 2 reduces to,

$$P^{\eta,\sigma}_{0+}[t^{\beta-1}\,{}_1E^{\gamma,q}_{1,\alpha,\beta}(\omega t^{\frac{\alpha}{1}})(x)]$$

$$=\frac{x^{\eta+\beta}}{(a(1-\sigma))^{\beta}}\quad\Gamma(\frac{\eta}{1-\sigma}+1)_1E^{\gamma,1}_{1,\alpha,\frac{\eta}{1-\sigma}+1+\beta}[\omega(\frac{x}{a(1-\sigma)})^{\alpha}]. \qquad (16)$$

Which is well known result earlier studied by Nair [11] and $E^{\gamma}_{\alpha,\beta}(z)$ is generalized Mittag-Leffler function studied by Prabhakar [12].

Theorem 3 Let $\eta,\gamma,\beta\in C$ and $R(1-\frac{\eta}{\sigma-1})>0$ Also, let $k,\sigma\in R$ with $\sigma>1$ and $p,q\in R^{+}$. then,

$$P^{\eta,\sigma}_{0+}[t^{\frac{\beta}{k}-1}\,{}_pE^{\gamma,q}_{k,\alpha,\beta}(\omega t^{\frac{\alpha}{k}})(x)]$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{(-a(\sigma-1))^{\frac{\beta}{k}}}P^{\frac{\eta}{\sigma-1}+1}\quad\Gamma(\frac{\eta}{\sigma-1}+1)\,{}_pE^{\gamma,q}_{k,\alpha,\frac{\eta}{\sigma-1}k+k+\beta}[\omega(\frac{x}{-a(\sigma-1)})^{\frac{\alpha}{k}}]. \qquad (17)$$

Proof Using equation (5) and (7), we have,

$$P^{\eta,\sigma}_{0+}[t^{\frac{\beta}{k}-1}\,{}_pE^{\gamma,q}_{k,\alpha,\beta}(\omega t^{\frac{\alpha}{k}})(x)]$$

$$=x^{\eta}\int_0^{\frac{x}{-a(\sigma-1)}}(1+\frac{a(\sigma-1)t}{x})^{\frac{\eta}{-\sigma-1}}t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{(\omega t^{\frac{\alpha}{k}})^n}{n!}dt.$$

$$=x^{\eta}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{\omega^n}{n!}\int_0^{\frac{x}{-a(\sigma-1)}}(1+\frac{a(\sigma-1)t}{x})^{\frac{\eta}{\sigma-1}}t^{\frac{n\alpha}{k}+\frac{\beta}{k}-1}dt.$$

Put $\frac{-a(\sigma-1)t}{x}=v$ then,

$$=x^{\eta}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{\omega^n}{n!}\int_0^1(1-v)^{\frac{\eta}{\sigma-1}}(\frac{x}{-a(\sigma-1)})^{\frac{n\alpha}{k}+\frac{\beta}{k}-1}v^{\frac{n\alpha}{k}+\frac{\beta}{k}-1}\frac{-x}{a(\sigma-1)}dv$$

. Using definition of beta function,

$$=x^{\eta}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)}\frac{\omega^n}{n!}(\frac{x}{-a(\sigma-1)})^{\frac{n\alpha}{k}+\frac{\beta}{k}}\frac{\Gamma(\frac{\eta}{\sigma-1}+1)\Gamma(\frac{n\alpha}{k}+\frac{\beta}{k})}{\Gamma(\frac{\eta}{\sigma-1}+\frac{n\alpha}{k}+\frac{\beta}{k})}$$

. Using the relation between p-k Gamma Function and classical Gamma Function [5](equation 2.19), we have

$$=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{\sigma-1}+1)}{(-a(\sigma-1))^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{nq,k}\,p^{\frac{\eta}{\sigma-1}+1}}{{}_p\Gamma_k((\frac{\eta}{\sigma-1})k+k+\beta+n\alpha)}(\omega(\frac{x}{-a(\sigma-1)})^{\frac{\alpha}{k}})$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{(-a(\sigma-1))^{\frac{\beta}{k}}}P^{\frac{\eta}{\sigma-1}+1}\quad\Gamma(\frac{\eta}{\sigma-1}+1)\,{}_pE^{\gamma,q}_{k,\alpha,\frac{\eta}{\sigma-1}k+k+\beta}[\omega(\frac{x}{-a(\sigma-1)})^{\frac{\alpha}{k}}]$$

This establish result(17).

Corollary 4.4 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{\sigma-1}>-1$ Also, let $\sigma\in R$ with $\sigma>1$, $p,q\in R^{+}$ and $p=k$ then theorem 3 reduces to,

$$P^{\eta,\sigma}_{0+}[t^{\frac{\beta}{k}-1}\,{}_kE^{\gamma,q}_{k,\alpha,\beta}(\omega t^{\frac{\alpha}{k}})(x)]$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{(-a(\sigma-1))^{\frac{\beta}{k}}}k^{\frac{\eta}{\sigma-1}+1}\quad\Gamma(\frac{\eta}{\sigma-1}+1)\,{}_kE^{\gamma,q}_{k,\alpha,\frac{\eta}{\sigma-1}k+k+\beta}[\omega(\frac{x}{-a(\sigma-1)})^{\frac{\alpha}{k}}].\qquad(18)$$

Which is new result.

Corollary 4.5 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{\sigma-1}>-1$ Also, let $\sigma\in R$ with $\sigma>1$, $p,q\in R^{+}$ and $p=k=1$ then theorem 3 reduces to,

$$P^{\eta,\sigma}_{0+}[t^{\beta-1}\,{}_1E^{\gamma,q}_{1,\alpha,\beta}(\omega t^{\frac{\alpha}{1}})(x)]$$

$$=\frac{x^{\eta+\beta}}{(-a(\sigma-1))^{\beta}}\quad\Gamma(\frac{\eta}{\sigma-1}+1)\,{}_1E^{\gamma,q}_{1,\alpha,\frac{\eta}{\sigma-1}+1+\beta}[\omega(\frac{x}{-a(\sigma-1)})^{\alpha}].\qquad(19)$$

Which is new result.

Corollary 4.6 Let $\eta,\gamma,\beta\in C$ and $\frac{\eta}{\sigma-1}>-1$ Also, let $\sigma\in R$ with $\sigma>1$, $p,q\in R^{+}$, $p=k=1$, and $q=1$, then theorem 3 reduces to,

$$P^{\eta,\sigma}_{0+}[t^{\beta-1}\,{}_1E^{\gamma,q}_{1,\alpha,\beta}(\omega t^{\frac{\alpha}{1}})(x)]$$

$$=\frac{x^{\eta+\beta}}{(-a(\sigma-1))^{\beta}}\quad\Gamma(\frac{\eta}{\sigma-1}+1)\,{}_1E^{\gamma,1}_{1,\alpha,\frac{\eta}{\sigma-1}+1+\beta}[\omega(\frac{x}{-a(\sigma-1)})^{\alpha}].\qquad(20)$$

This is well known result earlier studied by Nair [11].

5 Order and Type of the p-k Mittag-Leffler Function

In this section, we present some caracteristics of ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ function.We will show that ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ function is an entire function and we give the order and type of this function.

Theorem 4 The ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ function is an entire function of order η and type μ is given by,

$$\eta = \frac{k}{Re(\alpha)} and \mu = [\eta p^q e^{Re(\frac{\alpha}{k} ln(\frac{\alpha}{k}))\eta}]^{-1}. \tag{21}$$

Moreover, for all $\varepsilon > 0$, the next asympotic estimate holds:

$$_pE_{k,\alpha,\beta}^{\gamma,q}(z) < e^{(\mu+\varepsilon)|z|^\eta}. \tag{22}$$

with η, μ sufficiently large for $|z| \geq r_0$, $r_0 = r_0(\varepsilon) > 0$.

Proof The radius of convergence of p-k Mittag-Leffler function will be called R. Using [5] (equation (2.19) and equation (2.22))and the asymptotic expansions for the gamma function(see example [3]); and the asymptotic Stirling's formula:

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} e^{-z}[1+O(z-1)](|arg(z)| < \pi; |z| \to \infty); \tag{23}$$

in particular,

$$n! = (2\pi n)^{\frac{1}{2}} n^n e^{-n}[1+O(n^{-1})](n \in N; n \to \infty); \tag{24}$$

and the quotient expansion of two Gamma functions at infinity:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}[1+O(z^{-1})](|arg(z)+a| < \pi; |z| \to \infty). \tag{25}$$

Rewriting the equation (5) in following way,

$$_pE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{_p(\gamma)_{nq,k}}{_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} c_n z^n. \tag{26}$$

Since,

$$R = \limsup_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|; \tag{27}$$

using equations (23),(24) and (25), we obtain,

$$\left|\frac{c_n}{c_{n+1}}\right| = \left|\frac{_p(\gamma)_{nq,k}}{_p\Gamma_k(n\alpha+\beta)n!} \frac{_p\Gamma_k(\alpha(n+1)+\beta)(n+1)!}{_p(\gamma)_{(n+1)q,k}}\right|$$

$$= \left|\frac{_p\Gamma_k(\gamma+nqk)}{_p\Gamma_k(\gamma)\ _p\Gamma_k(n\alpha+\beta)n!} \frac{_p\Gamma_k(\gamma)\ _p\Gamma_k(n\alpha+\alpha+\beta)(n+1)!}{_p\Gamma_k(\gamma+(n+1)qk)}\right|$$

$$= \left|\frac{p^{\frac{\gamma+nqk}{k}}}{k} \frac{\Gamma(\frac{\gamma+nqk}{k})k(n+1)}{p^{\frac{n\alpha+\beta}{k}}\Gamma(\frac{n\alpha+\beta}{k})} \frac{p^{\frac{n\alpha+\alpha+\beta}{k}}}{k} \frac{\Gamma(\frac{n\alpha+\alpha+\beta}{k})k}{p^{\frac{\gamma+(n+1)qk}{k}}\Gamma(\frac{\gamma+(n+1)qk}{k})}\right|$$

$$= (n+1)|p^{\frac{\alpha}{k}-q}| \left|\frac{\Gamma(\frac{n\alpha}{k}+\frac{\alpha}{k}+\frac{\beta}{k})}{\Gamma(\frac{\alpha n}{k}+\frac{\beta}{k})}\right| \left|\frac{\Gamma(\frac{\gamma}{k}+nq)}{\Gamma(nq+\frac{\gamma}{k}+q)}\right|$$

$$\approx | p^{\frac{\alpha}{k}-q} \| (\frac{\alpha n}{k})^{\frac{\alpha}{k}} \| (nq)^{-q} | \to \infty. \tag{28}$$

Thus, the p-k Mittag-Leffler function is an entire function.
To find the order η and the type μ we applied the next definitions:

$$\eta = \limsup_{n\to\infty} \frac{nlnn}{ln(\frac{1}{|c_n|})}. \tag{29}$$

$$en\mu = \limsup_{n\to\infty} (n|c_n|^{\frac{\eta}{n}}). \tag{30}$$

Using the relations (23),(24),(25), and (26), we have,

$$\frac{1}{|c_n|} = |\frac{{}_p\Gamma_k(\gamma)\,{}_p\Gamma_k(n\alpha+\beta)n!}{{}_p\Gamma_k(\gamma+nqk)}|$$

$$= k^{-1}n!|\, p^{n(\frac{\alpha}{k}-q)+\frac{\beta}{k}} \| \Gamma(\frac{n\alpha}{k}+\frac{\beta}{k}) \| \Gamma(\frac{\gamma}{k}+nq)|^{-1} | \Gamma(\frac{\gamma}{k})|$$

$$\approx k^{-1}((2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}) |\, p^{n(\frac{\alpha}{k}-q)+\frac{\beta}{k}} | (\sqrt{2\pi}\,|(n\frac{\alpha}{k})^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{1}{2}} \| e^{-\frac{n\alpha}{k}}|)$$

$$\times (\sqrt{2\pi}\,|(nq)^{\frac{\gamma}{k}+n-\frac{1}{2}} \| \Gamma(\frac{\gamma}{k}|)^{-1} | \Gamma(\frac{\gamma}{k})|$$

$$\approx k^{-1}\sqrt{2\pi}\,|\, p^{n(\frac{\alpha}{k}-q)} \| (\frac{\alpha}{k})^{\frac{n\alpha}{k}} \| n^{n\frac{\alpha}{k}} \| \Gamma(\frac{\alpha}{k})| e^{-nRe(\frac{\alpha}{k})}$$

$$= e^{\frac{1}{2}ln(\frac{2\pi}{k})+ln|\Gamma(\frac{\alpha}{k})|+Re[n(\frac{\alpha}{k}-q)lnp+n\frac{\alpha}{k}ln\frac{\alpha}{k}+n\frac{\alpha}{k}lnn-n\frac{\alpha}{k}]}. \tag{31}$$

Hence, using (31), by application of (29), we have,

$$\frac{1}{\eta} = \limsup_{n\to\infty} \frac{\frac{1}{2}ln(\frac{2\pi}{k})+ln|\Gamma(\frac{\alpha}{k})|+Re[n(\frac{\alpha}{k}-q)lnp+n\frac{\alpha}{k}ln\frac{\alpha}{k}+n\frac{\alpha}{k}lnn-n\frac{\alpha}{k}]}{nlnn}$$

,

$$\frac{1}{\eta} = \frac{Re(\alpha)}{k}. \tag{32}$$

which is the wanted result in equation(21).

Similarly, on substituting (31) into (30), this yields,

$$e\eta\mu = \limsup_{n\to\infty} [n|c_n|^{\frac{\eta}{n}}]$$

,

$$= \limsup_{n\to\infty} [e^{lnn-\frac{1}{2}ln(\frac{2\pi}{k})-ln|\Gamma(\frac{\alpha}{k})|-Re[n(\frac{\alpha}{k}-q)lnp+n\frac{\alpha}{k}ln\frac{\alpha}{k}+n\frac{\alpha}{k}lnn-n\frac{\alpha}{k}]\frac{k}{nRe(\alpha)}}]$$

$$= p^{-q}e^{-Re(\frac{\alpha}{k}ln(\frac{\alpha}{k}))\frac{k}{Re(\alpha)}}. \tag{33}$$

Finally, we obtain that,

$$\mu = [\eta p^{q} e^{Re(\frac{\alpha}{k} ln(\frac{\alpha}{k}))\eta}]^{-1}. \qquad (34)$$

special case: If we substitute q=1 and replace μ and η by σ and ρ respectively in equation (34), then,

$$\sigma = [\rho p e^{Re(\frac{\alpha}{k} ln(\frac{\alpha}{k}))\rho}]^{-1}. \qquad (35)$$

which is known result given by [10](equation(III.15)).

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