Total Prime Labeling of Some Cycle and Path Related Graphs

S. Meena[1] and A. Ezhil[2]

[1]PG and Research Department of Mathematics, Government Arts College, C.Mutlur, Chidambaram 608 001. Tamilnadu, India.

[2] PG and Research Department of Mathematics, Thiru Kolanjiappar Government Arts College, Vridhachalam 606 001, Tamilnadu, India.

Abstract:

Let G = (V, E) be a graph with 'p' vertices and 'q' edges. A bijection f : V (G) → { 1,2,3,…………,p } is said to be prime labeling if for each edge e = uv , the labels assigned to u and v are relatively prime. A bijection f : V (G) U E (G) → {1,2,3……..(p+q)} is said to be a total prime labeling if,

(i) For each edge e = uv, the labels assigned to u and v are relatively prime.
(ii) For each vertex of degree at least two, the greatest common divisor of the labels of the incident edges is one.

In this paper we investigated the total prime labeling of the one side triangular graph (t_n), graph ($C_n . P_m$), graph (C_n (m)), graph m copies of C_n and comb related graphs.

Keywords: Labeling , prime labeling, total prime labeling.

Introduction:

Here, we consider only the graphs which are finite, simple and undirected graphs. A graph G = (V (G), E (G)) where V (G) denotes the vertex set and E (G) denotes the edge set. The order and size of the graph G are denoted by 'p' and 'q' respectively. For all other terminology and notations in graph theory, we follow Harary [1]. The notion of prime labeling was introduced by Rojer Entringer and was discussed in a paper by Tout [2] and vertex prime labeling was discussed in a paper by Deretesky [3].

Graph labeling where the vertices and edges are assigned real values with satisfying some conditions. Prime labeling and vertex prime labeling are already introduced. Combining these two results a new labeling called a total prime labeling was defined by Kala and Ramasubramanian [4] and they proved that the graphs Cycle, Path, Star, Bistar, Comb, Helm are total prime labeling graphs. Two integers are said to be relatively prime means the greatest common divisor is one. By the reference we proved that the graphs wheel, gear, carona, triangular book, double comb and planter graphs are total prime graphs[5].

Now we investigated the one-side triangular graph, graph m copies of C_n, graph $C_n . P_m$, graph C_n(m) ,subdivision of pendent edges of comb and double comb are all total prime graphs.

Definition 1.1:

Let G = (V, E) be a graph with 'p' vertices, A bijection f: V (G) → {1,2,3,……p} is said to be as prime labeling if for each edge e = uv the labels assigned to U and V are relatively prime. A graph which admits prime labeling is called prime graph.

Definition 1.2:

Let G = (V, E) be a graph with 'p' vertices and 'q' edges. A bijection f: E (G) → {1,2,……q} is said to be vertex prime labeling if for each vertex of degree at least two the greatest common divisor of the labels on its incident edges is one.

Definition 1.3:

Let G = (V, E) be a graph with 'p' vertices and 'q' edges. A bijection f: VUE → {1,2,3,….. (p+q)} is said to be total prime labeling if,

(i) for each edge e = uv, the labels assigned to U and V are relatively prime.
(ii) for each vertex of degree at least two, the gcd of the labels on the incident edge is one.

A graph which admits total prime labeling is called total prime graph.

Definition 1.4:

A subdivision graph S (G) of a graph G is a graph that can be obtained from G by subdividing each edge of G exactly once.

Definition 1.5:

The path P_n has n vertices and n-1 edges.

Definition 1.6:

A graph obtained by attaching a single pendent edges to each vertex of a path $P_n = v_1 v_2 v_3 \ldots v_n$ is called a comb.
Main Results:

Theorem 2. 1: The one side triangular graph t_n is total prime graph (n is even).

Proof:

Let $V(t_n) = \{ v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1} \}$
$E(t_n) = \{ v_i v_{i+1} / 1 \le i \le 2n \} \cup \{ v_{2i-1} v_{2i+1} / 1 \le i \le n \}$

Here total number of vertices p is 2n+1 and the total number of edges q is 3n.Hence p + q = 5n+1.

Define a labeling f: VUE → { 1, 2, 3,……….. (5n + 1 } is defined by
$f(v_i) = I;\ 1 \le i \le 2n + 1$
$f(e_i) = 2n + 1 + I;\ 1 \le i \le 3n$

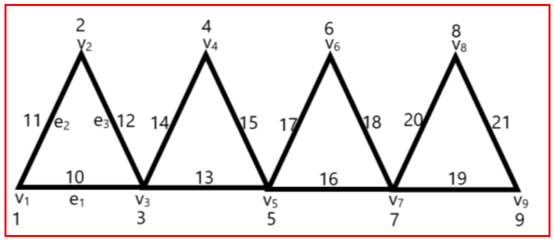
According to this pattern,

(i) $\gcd \{ v_i, v_{i+1} / 1 \le i \le 2n \} = \gcd \{ i, i+1 \} = 1$ for $1 \le i \le n$
(ii) $\gcd \{ v_{2i-1}, v_{2i+1} \} = \gcd \{ 2i-1, 2i+1 \} = 1$ for $1 \le i \le n$
(iii) gcd { all the edges incident with v_1 } = gcd { 2n+2, 2n+3 } = 1
(iv) gcd { all the edges incident with v_{2n+1} } = gcd { 5n-1, 5n } = 1
(v) gcd { all the edges incident with v_{2i} } = gcd { 2n+3i, 2n+3i+1 } = 1 for $1 \le i \le n$
(vi) gcd { all the edges incident with v_{2i+1} } = gcd { 2n+3i-1, 2n+3i+1, 2n+3i+2, 2n+3i+3 } = 1 for $1 \le i \le n - 1$.

Thus for each edge e = u v, where u and v are relatively prime and the gcd of each vertices of degree at least two, all the incident edges is one. Therefore, the graph (t_n) is a total prime graph.

Example:

Total prime graph of one side triangular graph t_4 (n even).



Theorem 2.2: The one side triangular graph t_n is total prime graph (n is odd).

Proof:

Let V (G) = { v_1, v_2, ……….v_n , v_{n+1}, …….v_{2n}, v_{2n+1} } and
E (G) = { v_i v_{i+1} / $1 \le i \le 2n$ } u { v_{2i-1} v_{2i+1} / $1 \le i \le n$}

Here total number of vertices p is 2n+1 and the total number of edges q is 3n.Hence p + q = 5n+1.

Define a labeling f: VUE → { 1, 2, 3 ,……….. (5n+1)}

$f(v_i) = i$; $1 \le i \le 2n+1$

$f(e_i) = 2n+1+i$; $1 \le i \le 3n-2$

$f(e_i) = 2n+2+i$; $i = 3n-1$

$f(e_i) = 2n+i$; $i = 3n$

According to this pattern,

(i) gcd { v_i, v_{i+1} } = gcd { i, i+1 } = 1 for $1 \le i \le 2n$

(ii) gcd { v_{2i-1} , v_{2i+1} } = gcd { 2i-1, 2i+1 } = 1 for $1 \le i \le n$

(iii) gcd { all the edges incident with v_1 } = gcd { 2n+2, 2n+3 } = 1

(iv) gcd { all the edges incident with v_{2n} } = gcd { 5n-1, 5n } = 1

(v) gcd { all the edges incident with v_{2i} } = gcd { 2n+3i, 2n+3i+1 } = 1 for $1 \le i \le n-1$

(vi) gcd { all the edges incident with v_{2n+1} } = gcd { 5n-1, 5n }= 1

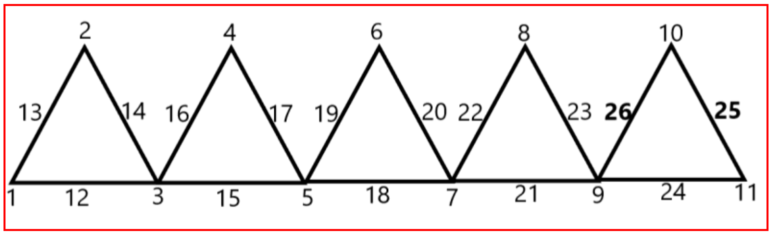
(vii) gcd { all the edges incident with v_{2i+1} } = gcd { 2n+3i-1, 2n+3i+1, 2n+3i+2, 2n+3i+3 } = 1 for $1 \le i \le n-2$

(viii) gcd { all the edges incident with v_{2n-1} } = gcd { 5n-4, 5n-2, 5n-1, 5n+1 } = 1.

Thus for each edge e = u v, where u and v are relatively prime and the gcd of each vertices of degree at least two, all the incident edges is one. Therefore, the triangular graph (t_n), (n is odd) is a total prime graph.

Example:

Total prime graph of one side triangular graph t_4 (n odd).



Theorem 2.3: The graph m copies of C_n is a total prime graph (for all m , n)

Proof:

Let C_n is a cycle with n vertices and n edges. Here G is m copies of C_n where C_n are joined by a single path.

Let V (G) = { v_{11}, v_{12}, ……….v_{1n}, v_{21}, v_{22}, ……….v_{2n1} ,……….v_{m1}, v_{m2}, ……….v_{mn} } and
E (G) = { v_{ij} v_{ij+1} / 1 ≤ i ≤ m, 1 ≤ i ≤ n } U { v_{in} v_{i1} / 1 ≤ i ≤ m } U { v_{i1} $v_{(i+1)1}$ / 1 ≤ i ≤ m-1}.

Hence, the total number of vertices p is mn and the total number of edges q is nm+m-1. Hence p + q = 2mn+m-1.

Define a labeling f:VUE →{ 1,2,……(2mn+m-1) } as follows

f (v_{ij}) = n (i-1) + j ; 1 ≤ i ≤ m, 1 ≤ i ≤ n
f (e_{ij}) = mn +(i-1) n + i+(j-1) ; 1 ≤ i ≤ m, 1 ≤ i ≤ n.

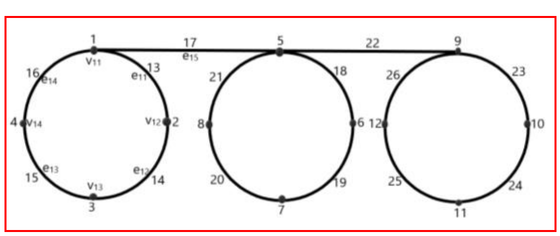
According to this pattern,

(i) gcd { v_{ij}, v_{ij+1} } = gcd { n(i-1) +1 }, { n(i-1) + j+1 } = 1 for 1 ≤ i ≤ m, 1 ≤ j ≤ n-1
(ii) gcd { v_{ij}, v_{ij+1} } = gcd { n (i-1) + j, n (i-1) + j + 1 } = 1 for1 ≤ j ≤ n - 1
(iii) gcd { v_{in}, v_{ij} } = gcd { n(i-1) + j, n (i-1) + j + 1 } = 1 for 1 ≤ i ≤ m
(iv) gcd { all the edges incident with v_{ij} } = gcd { mn + (i-1) n + i + j+ 1, mn + (i-1) n +i +j+1 } = 1 for 1 ≤ i ≤ m, 1 ≤ i ≤ n - 1
(v) gcd { all the edges incident with v_{ii} } = gcd { e_{11}, e_{1n} , e_{1n+1} } = { mn + 1, mn + n , mn + n + 1 } = 1
(vi) gcd { all the edges incident with v_{m1} } = gcd { e (m-1)(n+1), e_{m1}, e_{mn} } = gcd { mn + n(m-1) + n +m+1, mn +(m-1)+1, mn + n(m-1)+m+n+1 , mn +(m-1) + l , mn + n(m-1)+ m+ n- 1 } = 1

Thus for each edge e = u v, where u and v are relatively prime and the gcd of each vertices of degree at least two, all the incident edges is one. Therefore, the graph m copy of C n is a total prime graph.

Example:

Total prime graph of three copies of c_4.



Theorem 2.4: The graph obtained by subdivision of pendent edges of a comb $P_n. K_1$ is a total prime labeling graphs.

Proof:

Let $P_n.K_1$ be a comb obtained from the path $P_n = v_1 v_2 \ldots\ldots. v_n$ and by joining a vertex u_i to v_i, $1 \leq i \leq n$, where u_i are the pendent vertices adjacent to v_i.

Let

$V(P_n .K_1) = \{ v_1, v_2, \ldots\ldots\ldots\ldots v_n, u_1, u_2, \ldots\ldots\ldots\ldots u_n \}$

$E(P_n .K_1) = \{ v_i u_i \ / \ 1 \leq i \leq n \} \cup \{ v_i v_{i+1} \ / \ 1 \leq i \leq n-1 \}$

Now the pendent edges are subdivided by two edges. The subdivision of pendent edges of a comb $Pn.K_1$ is derived and $S(P_n .K_1)$. The new vertices are $u_1', u_2', \ldots.. u_n'$.

Let $V[S(P_n .K_1)] = \{ v_1, v_2, \ldots\ldots. v_n, u_1, u_2, \ldots\ldots. u_n, u_1', u_2' \ldots\ldots. u_n' \}$

$E[S(P_n .K_1)] = \{ v_i u_i \ / \ 1 \leq i \leq n \} \cup \{ u_i u_{i'} \ / \ 1 \leq i \leq n \} \cup \{ v_i v_{i+1} \ / \ 1 \leq i \leq n - 1 \}$ and

The total number of vertices p =3n. The total number of edges q = 3n-1. Hence p+q=6n-1.

Define a bijection

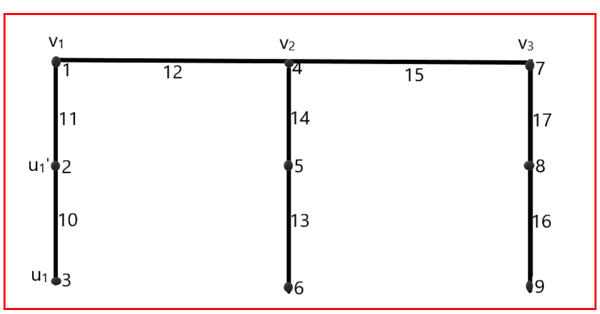
$f: V \cup E \rightarrow \{ 1,2,3,\ldots\ldots.(6n-1) \}$ by
$f (v_i) = 3i - 2 \ ; \ 1 \leq i \leq n$
$f (u_i') = 3i - 1 \ ; \ 1 \leq i \leq n$
$f (u_i) = 3i \ ; \ 1 \leq i \leq n$
$f (e_i) = 3n + I \ ; \ 1 \leq i \leq 3n-1.$

According to this pattern

(i) $\gcd \{ v_i , u_i' \} = \gcd \{ 3i-2, 3i-1 \} = 1$ for $1 \leq i \leq n$
(ii) $\gcd \{ u_i', u_i \} = \gcd \{ 3 i-1, 3i \} = 1$ for $1 \leq i \leq n$
(iii) $\gcd \{ v_i , v_{i+1} \} = \gcd \{ 3i-2, 3i+1 \} = 1$ for $1 \leq i \leq n-1$
(iv) gcd { all the edges incident with u_i' } = $\gcd \{ 3n+3i-2, 3n+3i-1 \} = 1$ for $1 \leq i \leq n$.
(v) gcd { all the edges incident with v_1 } = $\gcd \{ 3n+2, 3n+3 \} = 1$.
(vi) gcd { all the edges incident with v_n } = $\gcd \{ 6n-3, 6n-1 \} = 1$.
(vii) gcd { all the edges incident with v_i } = $\gcd \{ 2n+3i, 3n+3i \} = 1$ for $2 \leq i \leq n - 1$.

Thus for each edge e = u v, where u and v are relatively prime and the gcd of each vertices of degree at least two, all the incident edges is one. Therefore, the graph obtained by subdivision of the edges $S(P_n.K_1)$ is a total prime graph.

Example:
Total prime graph of $S(P_3.K_1)$



Theorem 2.5: The graph obtained by subdivision of pendent edges of a double comb $P_n .K_{1,2}$ is total prime graph.

Proof:

The double comb is obtained from the path joining a double side pendent edges.

Let V(G) = { v_1, v_2, ...v_n, w_1, w_2, ...w_n, ...u_1, u_2, ...u_n } and
E(G) = { v_iw_i / $1 \le i \le n$ } U { v_iu_i / $1 \le i \le n$ } U { v_iv_{i+1} / $1 \le i \le n-1$}

Now, subdividing the pendent edges of both sides of the comb by a single vertex, is denoted by $S(P_n.K_{1,2})$

Let V [S ($P_n.k_{1,2}$)] = { v_1 ,v_2, ...v_n, w_1, ...w_n, u_1 ,u_2 ,.......u_n, w_1' ,w_2', ...w_n', u_1' ,u_2', ...u_n' }
E [S ($P_n.k_{1,2}$)] = { v_iw_i' / $1 \le i \le n$ } U { $w_i'w_i$ / $1 \le i \le n$ } U { v_iu_i' / $1 \le i \le n$ } U { $u_i'u_i$ / $1 \le I \le n$ } U { v_iv_{i+1} / $1 \le i \le n$}.

The total number of vertices are p = 5n and the total number of edges are q = 5n-1. Hence p+q = 10n-1.

Define a labeling

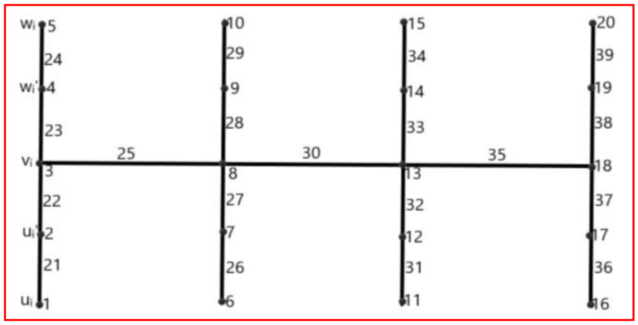
F: VUE → {1,2,........(10n-1)} by
f: (v_i) = 5 i – 2 ; $1 \le i \le n$
f: (u_i) = 5 i – 4 ; $1 \le i \le n$
f: (u_i ‘) = 5 i – 3 ; $1 \le i \le n$
f: (w_i') = 5 i – 1; $1 \le i \le n$
f: (w_i) = 5 i; $1 \le i \le n$
f: (e_i) = 5 n + i ; $i \le I \le 5n-1$

According to this pattern,

i) gcd { u_i , u_i' } = gcd { 5 i – 4, 5i -3} = 1 for $1 \le i \le n$
ii) gcd { u_i ‘ ,v_i } = gcd { 5 i – 3, 5i -2} = 1 for $1 \le i \le n$
iii) gcd { v_i , w_i' } = gcd { 5 i – 4, 5i -1} = 1 for $1 \le i \le n$
iv) gcd { w_i', w_i } = gcd { 5 i – 1, 5i } = 1 for $1 \le i \le n$
v) gcd { v_i , v_{i+1} } = gcd { 5 i – 2, 5i +3} = 1 for $1 \le i \le n-1$
vi) gcd { all the edges incident with w_i' } = gcd { 5 n + 5 i – 2 , 5 n + 5i - 1 } = 1 for $1 \le i \le n$
vii) gcd { all the edges incident with u_i' } = gcd { 4n + 5 i – 4 , 4n + 5i - 3} = 1 for $1 \le i \le n$
viii) gcd { all the edges incident with with v_1} = gcd { 4 n +2, 4 n +3, 4 n +5 } = 1
ix) gcd { all the edges incident with v_n }= gcd { 9 n +1 ,10 n -3, 10 n -2 } = 1
x) gcd { all the edges incident with v_{i+1} } = gcd { 6 n + 5 i – 4, 6 n+5i – 2, 6 n +5 i- 1, 6n + 5 i +1 } = 1 for $1 \le i \le n-1$

Thus for each edge e =uv, Where u and v are relatively prime and the gcd of each vertex of degree at least two all the incident edges is one. Therefore the graph obtained by subdivision of the edges of S ($P_n.K_{1,2}$) is a total prime graph.

Example: Total prime graph of S ($P_n.K_{1,2}$)



Theorem 2.6: The graph $C_n. P_m$ is a total prime graph(for all m,n)

Proof:

The graph $C_n. P_m$ contains n cycles with P_m paths such that the m paths are joined at each vertex of n cycles.

Let V ($C_n. P_m$) = { V_{11} V_{12} ….. V1n, V_{21} …. V_{nm} }

E ($C_n. P_m$) = { V_{ij} V_{ij+1} : $1 \le i \le n$, $1 \le j \le m -1$ } U { V_{i1}/ $1 \le i \le n$ }

The total number of vertices p = nm and the total number of edges q = nm.

Hence p +q -= 2nm

Define by a bijetion

f: VUE ⟶ { 1, 2, 3, …. 2mn} by

$f(v_{ij}) = m(i-1) + j$; $1 \le i \le n$, $1 \le j \le m$

$f(e_{ij}) = mn + m(i-1) + j$; $1 \le i \le n$, $1 \le j \le m$

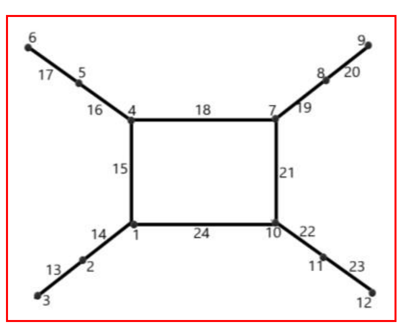
According to this pattern

i. gcd { v_{ij}, v_{ij+1} } = gcd { m i – 1) + j, m (i-1) +j+1} = 1 for $1 \le i \le n$, $1 \le i \le m$
ii. gcd { v_{i1}, $v_{(i+1)1}$ } = gcd {m(i-1)+1, mi } = 1 for $1 \le i \le n-1$, $1 \le j \le m$.
iii. gcd { all the edges incident with v_{ij} } = gcd { mn+m(i-1)+j, mn+m(i-1)+j+1 } = 1 for $1 \le i \le n$, $1 \le j \le m-1$.
iv. gcd { all the edges incident with v_{11} } = gcd { mn+m-1, mn+m, 2mn } = 1 for i=1.
v. gcd { all the edges incident with v_{n1} } = gcd { mn+m (n-2)+1, mn+m(n-1), mn+m (n-1)+1 } = 1.
vi. gcd { all the edges incident with $v_{(i+1)1}$ } = gcd { nm+m(i-1)+m, mn+mi+m-1, nm+mi+m } = 1 for $1 \le i \le (n-1)$.

Thus for each edge e = uv, where u and v are relatively prime and the gcd of each vertex of degree at least two all the incident edges is one. Therefore, the graph $C_n.P_m$ is a total prime graph.

Example:

Total prime graph $C_4.P_3$



Theorem 2.7: The graph C_n (m) is total prime graph (for all m , n)

Proof:

The graph C_n (m) is a graph which contains C_n cycles with m times such that the nth vertex all n cycles are joined by a single vertex (u)

Let $V(G) = \{ v_{11}, v_{12} \ldots v_{1(n-1)}, v_{21}, v_{22}, \ldots \ldots v_{2(n-1)}) v_{m1} \ldots v_{m(n-1)}, u \}$

$E(G) = \{ v_{ij} v_{ij+1} / 1 \leq i \leq m, 1 \leq j \leq n-2 \} \cup \{ uv_{i1} / 1 \leq i \leq m \} \cup \{ v_{i(n-1)} u / 1 \leq i \leq m\}$

Here, the total number of vertices P is m (n-1) and the total number of edges q is mn.
Hence p+ q = 2 mn – m+1.
Define a bijection, f: VUE ⟶ { 1, 2, …. (2mn – m + 1)} by

$f(v_{ij}) = (n-1)(i-1) + j + 1 ; 1 \leq i \leq m \quad 1 \leq j \leq n-1$

$f(e_{ij}) = m(n-1) + n(i-1) + j + 1 ; 1 \leq i \leq m \quad 1 \leq j \leq n-1$

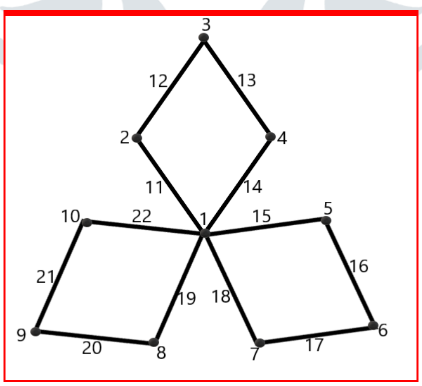
According to the pattern

i. gcd $\{ u, v_{i1} \}$ = gcd $\{ 1, (n-1) + (i-1) + j+1 \} = 1$ for $1 \leq i \leq m$
ii. gcd $\{ v_{i(n-1)}, u \}$ = gcd $\{ (n-1)(i-1)+j+1, 1 \} = 1$ for $1 \leq i \leq m$
iii. gcd $\{ v_{ij}, v_{ij+1} \}$ = gcd $\{ (n-1)(i-1) + j+1, (n-1)(i-1) + j+2 \} = 1$ for $1 \leq i \leq m$, $1 \leq j \leq n-2$
iv. gcd { all the edges incident with v_{ij} }= gcd $\{ m(n-1)+ n(i-1) + J+1, m(n-1) +n(i-1) + j+2 \}$ = 1 for $1 \leq i \leq m$, $1 \leq j \leq n-1$
v. gcd { all the edges incident with u } = gcd $\{ m(n-1)+ n(i-1) + J+1 \} = 1$ for $1 \leq i \leq m$, j = 1 and j = n.

Thus for each edge e = uv where u and v are relatively prime and for each vertex of degree at least two the gcd of all the incident edges is one. Therefore the graph C_n (m) is a total prime graph.

Example:

Total prime graph of C_4 (3)



References:
1. J.A. Bondy and U.S.R. Murthy. Graph Theory and Applications (North-Holland), New York, 1976.

2. J.A. Gallian. A Dynamic Survey of Graph Labelling. The Electronic Journal of Combinatorics, 18(2011), 147, DS6.
3. M. Ravi (a) Rama Subramanian, R. Kala. Total Prime Graph. International Journal of Computational Engineering Research (ijceronline.com), Vol 2. Issue 5, ISSN (2250-3005) online, Sep.2012, Page 1588-1592.
4. S. Meena and K. Vaithilingam. Prime Labeling for some Crown Related Graphs, International Journal of Scientific and Technology Research. Vol.2, Issue 3(2013), 92-95.
5. S. Meena and A. Ezhil. Total prime labeling of some graphs. International Journal of Research in Advent Technology. Vol.7, No.1, January 2019.
6. S. Meena and K. Vaithilingam. Prime Labeling for some Helm Related Graphs, International Journal of Innovative Research in Science, Engineering and Technology. 2(4) (2013).
7. S.K. Vaidya and Udayan M. Prajapati. Some Results on Prime and 'K-Prime' Labeling. Journal of Mathematics Research, 3(1) (2011) 66-75.
8. S.K. Vaidya and K.K. Kanani. Prime Labeling for some Cycle Related Graphs. Journal of Mathematics Research, Vol.2, No. 2010, 17-20.
9. S. Ashok Kumar and S.Maragathavalli. Prime Labeling of Special Graphs. IOSR, Journal of Mathematics, (IOSR-JM), Vol.11 (2015), 1-5.
10. S.K. Vaidya and Udayan M. Prajapati. Some new Results on Prime Graph. Open Journal of Discrete Mathematics, (2012), 99-104.