FUZZY CHROMATIC NUMBER OF FUZZY GRAPH FORMED FROM THE CARTESIAN PRODUCT OF FUZZY PATH AND FUZZY CYCLE

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Abstract: In this paper, we discussed the concept of chromatic number of a fuzzy graph. We established bounds for fuzzy chromatic number of fuzzy graph formed from the cartesian product of fuzzy path and fuzzy cycle.

Keywords: *Fuzzy Path, Fuzzy Cycle, Cartesian product of fuzzy path and fuzzy cycle, colouring, Fuzzy colouring and fuzzy chromatic number.*

1.1 Introduction

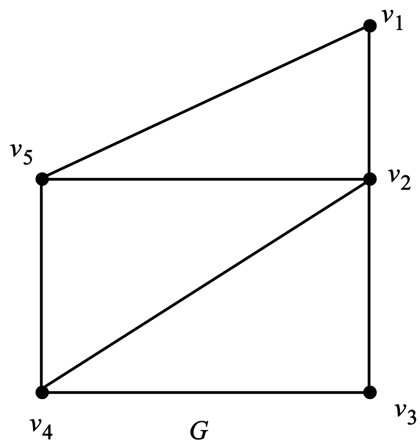
One of the most important properties of fuzzy graph model is fuzzy graph colouring. The fuzzy colouring of a fuzzy graph was defined by the authors in Eslachi and Onagh[12]. Then pourspasha[9] also introduced different approaches to colour the fuzzy graph.

Fuzzy dominator colouring and fuzzy chromatic number on Cartesian product of simple fuzzy graph was introduced by R.Muthuraj and Sasireka [10]. In this paper ,we deals with finding a fuzzy chromatic number of Cartesian product of fuzzy path and cycle.

Definition 1.1:

- A walk is called a **path** if all its vertices are distinct.
- A $v_0 - v_n$ walk is called closed if $v_0 = v_n$. A closed walk $v_0, v_1, v_2, ..., v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, ..., v_{n-1}$ are distinct is called a **cycle** of length *n*. The graph consisting of a cycle of length *n* is denoted by C_n.

Example:1.1

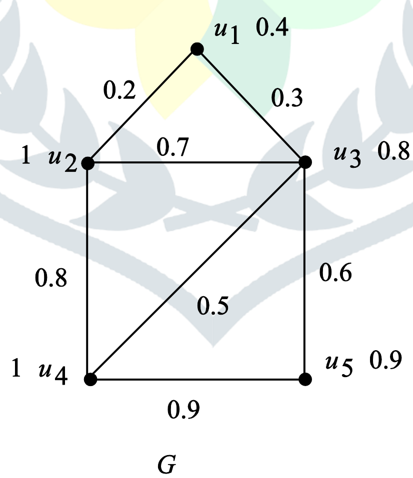


v_1, v_2, v_4, v_5 is a path.

Definition 1.2

Let *V* be a finite non empty set. The triple $G = (V, \sigma, \mu)$ is called a **fuzzy graph** on *V* where σ and μ are fuzzy sets on *V* and *E* respectively such that $\mu(u,v) \le \sigma(u) \Lambda \sigma(v)$ for all $u, v \in V$ and (u,v)€E. For fuzzy $G = (V, \sigma, \mu)$, the elements *V* and *E* are called set of vertices and set of edges of *G* respectively.

Example:1.2



Definition 1.3:

The **Cartesian product** $G = G_1 \times G_2 = (V, X)$ of graphs G_1 and G_2. Let $V = V_1 \times V_2$, and

$$X = \{(u, u_2), (u, v_2) \mid u \in V_1, (u_2, v_2) \in X_2\} \cup \{(u_1, w), (v_1, w) \mid w \in V_2, (u_1, v_1) \in X_1\}.$$

Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of X_i for every i=1,2. Define the fuzzy subsets $\sigma_1 \times \sigma_2$ of *V* and $\mu_1 \times \mu_2$ of *X* as follows:

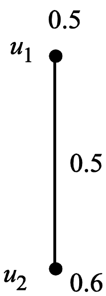
$(\sigma_1 \times \sigma_2)(u_1, u_2) = \min\{\sigma_1(u_1), \sigma_2(u_2)\}$ for all u_1 and $u_2 \in$ V

$(\mu_1 \times \mu_2)(u_1,u_2)(u,v_2) = \min\{\sigma_1(u_1), \mu_2(u_2,v_2)\}$ for all $u \in V_1$ and $(u_2,v_2) \in X_2$

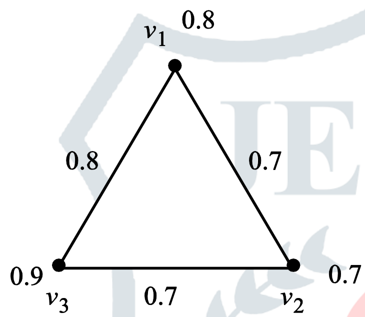
$(\mu_1 \times \mu_2)(u_1,w)(v_1,w) = \min\{\sigma_2(w), \mu_1(u_1,v_1)\}$ for all $w \in V_2$ and $(u_2, v_2) \in X_1$

Then the fuzzy graph G= $(\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ is said to be the Cartesian product of $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$.

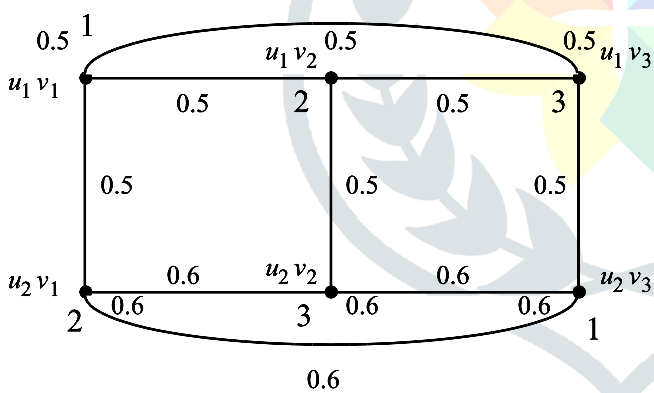
Example:



G



H



$G \times H$

2.PRELIMINARIES

Proposition2.1:

If G and H are fuzzy path on *m* and *n* vertices and Cartesian product of *G* and *H* is, $G \times H$ is a fuzzy graph on *mn* vertices. Then $\chi^f(G \times H) = 2$.

Proposition 2.2:

If G is a fuzzy path on m vertices and H is a fuzzy cycle on n vertices and the Cartesian product of G and H is $G \times H$ is a fuzzy graph on $(m \times n)$ vertices then $\chi^f(G \times H) = 2$, $\gamma(G \times H = 2)$, $\chi_d^f(G \times H) = 3$.

3: **Result on chromatic number of Cartesian product of fuzzy paths and cycles**

Let $G = (V_1, \sigma_1, \mu_1)$ be fuzzy path graph and $H = (V_2, \sigma_2, \mu_2)$ be a fuzzy cycle. The Cartesian product $G \times H = (V, X)$ of fuzzy path and cycle of *G* and *H*.

Where $V = V_1 \times V_2, X = \{(u,u_2),(u,v_2) / u \in V_1, (u_2,v_2) \in X_2\} \cup \{(u_1,w),(v_1,w) / w \in V_2, (u_1,v_1) \in X_1\}$

To prove this theorem we use independent fuzzy path and fuzzy cycles which does not have unique weak arc in fuzzy path and cycle of $G \times H$. Therefore, all arc would be strong arc.

Brook's gave an upper bound for the chromatic number in case of crisp graphs and stated that 'For a connected graph which is neither an odd cycle nor a complete graph, the chromatic number $\chi(G) \le \Delta(G)$'.

Theorem 3.1:

If $(G \times H)$ is a Cartesian product of fuzzy graph on $(m \times n)$ vertices where *G* is a fuzzy path graph of *m* vertices and *H* is a fuzzy cycle graph of *n* vertices, $\forall\ m, n \ge 2$ then

$$\chi^f(G \times H) = \begin{cases} 3 \text{ if } m \text{ is even, } n \text{ is odd} \\ 2 \text{ if } m \text{ is odd, } n \text{ is even} \\ 2 \text{ if } m \text{ and } n \text{ are even} \\ 3 \text{ if } m \text{ and } n \text{ are odd} \end{cases}$$

Proof:

Let *G* be a fuzzy path of *m* vertices and *H* be a fuzzy cycle of *n* vertices.The Cartesian product of *G* and *H* is $(G \times H)$ with *mn* vertices.Let $V(G) = \{u_1, ..., u_m\}$ be a vertex set of fuzzy path graph.

$V(H) = \{v_1, v_2, ..., v_n\}$ be a vertex set of fuzzy cycle graph.We assign the colours to the vertices as follows.

Case (i) Suppose that $m = 2k$, $n = 2k+1$ then to prove $\chi^f(G \times H) = 3, \forall\ k = 1,2,3,...$

When $m=2,\ n=3$ for k=1,We have $V(G)=\{u_1,u_2\}$, $V(H)=\{v_1,v_2,v_3\}$ and its Cartesian product is

$$V(U\times H)=\begin{Bmatrix} a_1=u_1v_1, a_2=u_1v_2, a_3=u_1v_3 \\ a_4=u_2v_1, a_5=u_2v_2, a_6=u_2v_3 \end{Bmatrix},$$

$$|V(G\times H)|=6$$

We assign colours to the vertices of $G\times H$ as follows.Assign colour 1 to vertices a_1 and a_6 as these vertices are non adjacent.Now assign colour 2 to the vertices a_2 and a_4 and assign colour 3 to vertices a_3 and a_5. Hence we required 3 colours for proper colouringof $(G\times H)$.Thus $\chi^f(G\times H)=3$.

Proceeding in this way, for the graph $(G\times H)$ having $(m\times n)$ vertices,when $m=2k$ and $n=2k+1$, $\forall\ k=1,2,3,\ldots$ i.e) The graph $(G\times H)$ contains $(2k)(2k+1)$ elements.

i.e) $$V(G\times H)=\begin{Bmatrix} u_1v_1,\ldots,u_1v_2,\ldots,u_1v_{2k+1} \\ u_2v_1,\ldots,u_2v_1,\ldots,u_3v_{2k+1} \\ u_3v_1,\ldots,u_3v_2,\ldots,u_3v_{2k+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ u_{2k}v_1 \quad u_{2k}v_2 \quad u_{2k}v_{2k+1} \end{Bmatrix}$$

$$|V(G\times H)|=(2k)(2k+1)=4k^2+2k$$

let $S_1=\{u_1v_1,\ldots,u_{2k}v_{2k+1}\}$ be the subset of $V(G\times H)$ and ,all the vertices of S_1 are assigned with Colour 1 since no two of its are adjacent.Let $S_2=\{u_2v_1,\ldots,u_{2k-2}v_{2k-3}\}$ be another subset of $V(G\times H)$ all the vertices of S_2 are assigned with colour 2 and the remaining vertices of $V(G\times H)$ is $S_3=\{u_1v_3,\ldots u_{2k}v_{2k}\}$ assigned with colour 3.Therefore we required 3 colours for fuzzy vertex proper colouring.Hence the fuzzy chromatic number of $(G\times H)$ is $\chi^f(G\times H)=3$.

Case (ii) Suppose that $m=2k+1,\ n=2k$ then to prove $\chi^f(G\times H)=2$, $\forall\ k=1,2,3,\ldots$

When $m=3,\ n=4$, We have $V(G)=\{u_1,u_2,u_3\}$ $V(H)=\{v_1,v_2,v_3,v_4\}$ and its Cartesian product is

$$V(G\times H)=\begin{Bmatrix} a_1=u_1v_1, a_2=u_1v_2, a_3=u_1v_3, a_4=u_1v_4 \\ a_5=u_2v_1, a_6=u_2v_2, a_7=u_2v_3, a_8=u_3v_4 \\ a_9=u_3v_1, a_{10}=u_3v_2, a_{11}=u_3v_3, a_{12}=u_3v_4 \end{Bmatrix}$$

$$|V(G\times H)|=12$$

Assign colour 1 to vertex set $\{a_1, a_3, a_6, a_8, a_9, a_{11}\}$.Since these vertices are non adjacent..

Now assign colour 2 to the vertices $\{a_2, a_4, a_5, a_7, a_{10}, a_{12}\}$ since these vertices are non adjacent.

Since a_1 and a_2 are adjacent vertices we assigned with different colour 1 and 2.

Hence we required colours for proper colouringof $(G\times H)$.Thus $\chi^f(G\times H)=2$

In general, let $(G\times H)$ be a graph having $(m\times n)$ Take $m=2k+1$ and $n=2k\ \forall\ k=1,2,3,\ldots$ The graph $(G\times H)$ contains $(2k+1)(2k)$ elements.

i.e) $V(G\times H)=\begin{Bmatrix} u_1v_1, u_1v_2, \ldots, u_1v_{2k} \\ u_2v_1, \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ u_{2k+1}v_1, \ldots, u_{2k+1}v_{2k} \end{Bmatrix}$

$$|V(G\times H)|=(2k+1)(2k)=4k^2+2k$$

Let $S_1=\{u_1v_1, u_1v_3, \ldots, u_{2k+1}v_{2k}\}$ be the subset of $V(G\times H)$ and by definition of proper colouring ,all the vertices of S_1 are assigned with colour 1 since no two of its are adjacent ,let $S_2=\{u_1v_2, u_1v_4, \ldots, u_{2k+1}v_{2k-1}\}$ be another subset of $V(G\times H)$,all the verticesof S_2 are assigned with colour 2.Therefore we required 2 colours for fuzzy vertex proper colouring

Hence the fuzzy chromatic number of $(G\times H)$ is $\chi^f(G\times H)=2$.

Case (iii) If $m=2k$, $n=2k$ then to prove $\chi^f(G\times H)=2$, $\forall\ k=1,2,3,\ldots$ When $m=2,\ n=4$

We have $V(G)=\{u_1,u_2\}$, $V(H)=\{v_1,v_2,v_3,v_4\}$ and its Cartesian product is

$$V(G\times H)=\begin{Bmatrix} a_1=u_1v_1, a_2=u_1v_2, a_3=u_1v_3, a_4=u_1v_4 \\ a_5=u_2v_1, a_6=u_2v_2, a_7=u_2v_3, a_8=u_3v_4 \end{Bmatrix}$$

$$|V(G\times H)|=8$$

We assign colours to the vertices of $G\times H$ as follows. Assign colour 1 to vertices $\{a_1, a_3, a_6, a_8\}$ and assign colour 2 to the vertices $\{a_2, a_4, a_5, a_7\}$

Since a_1 and a_2 are adjacent vertices we assigned with different colour 1 and 2.Hence we required 2 colours for proper colouringof $(G\times H)$.Thus $\chi^f(G\times H)=3$.

In general, let $(G \times H)$ be a graph having $(m \times n)$ with $m = 2k$ and $n = 2k \ \forall \ k = 1, 2, 3, \ldots$

The graph $(G \times H)$ contains $(2k)(2k)$ elements.

Let $V(G \times H) = \begin{Bmatrix} u_1v_1, & u_1v_2, \ldots, & u_1v_{2k} \\ u_2v_1, & u_2v_2, \ldots, & u_2v_{2k} \\ \vdots & & \\ u_{2k}v_1, & u_{2k}v_k, \ldots, u_{2k}v_{2k} & \end{Bmatrix}$

$|V(G \times H)| = (2k)(2k) = 4k^2$

Let us colour the vertices of the graph as follows.

Let $S_1 = \{u_1v_2, \ldots, u_2v_2, \ldots, u_{2k}v_{2k}\}$ be the subset of V$(G \times H)$ and by definition of proper colouring all the vertices of S_1 are assigned with colour 1 since no two of them are adjacent.Let $S_2 = \{u_2v_1, \ldots, u_{2k}v_{2k-1}\}$ be the subset of $V(G \times H)$,all the vertices of S_2 are assigned with colour 2 since no two of it is adjacent.Therefore we required 2 colour for fuzzy vertex proper colouring,

Hence the fuzzy chromatic number of $(G \times H)$ is $\chi^f(G \times H) = 2$.

Case (iv) Suppose that $m = 2k+1, \ n = 2k+1$ then to prove $\chi^f(G \times H) = 3, \forall \ k = 1, 2, 3, \ldots$

When $m = 3, \ n = 3$, We have $V(G) = \{u_1, u_2, u_3\}$, $V(H) = \{v_1, v_2, v_3\}$ and its Cartesian product is

$$V(G \times H) = \begin{Bmatrix} a_1 = u_1v_1, a_2 = u_1v_2, a_3 = u_1v_3, a_4 = u_2v_1, a_5 = u_2v_2 \\ a_6 = u_2v_3, a_7 = u_3v_1, a_8 = u_3v_2, a_9 = u_3v_3 \end{Bmatrix}$$

$|V(G \times H)| = 9$

We assign colour to the vertices of $(G \times H)$ as follows.

Assign colour 1 to vertices $\{a_1, a_6, a_8\}$ since these vertices are non adjacent so we assign same colour.Now assign colour 2 to the vertices $\{a_2, a_4\}$ since these vertices are non-adjacent so we assign same colour.Hence we required 3 colours for proper colouringof $(G \times H)$.Thus $\chi^f(G \times H) = 3$.

Proceeding in this way, In general, let $(G \times H)$ be a graph having $(m \times n)$ take $m = 2k+1$ and $n = 2k+1$ $\forall \ k = 1, 2, 3, \ldots$ The graph $(G \times H)$ contains $(2k+1)(2k+1)$ elements.

Let $V(G \times H) = \begin{Bmatrix} u_1v_1, & u_1v_2, & \ldots, u_1v_{2k+1} \\ u_2v_1, & u_2v_2, & \ldots, u_2v_{2k+1} \\ \vdots & & \\ u_{2k+1}v_1, & u_{2k+1}v_2, & \ldots, u_{2k+1}v_{2k+1} \end{Bmatrix}$

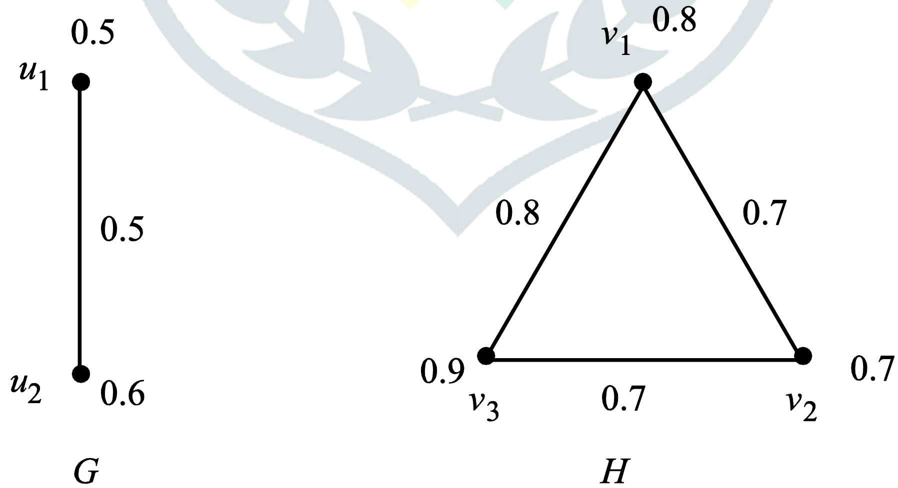
$|V(G \times H)| = (2k+1)(2k+1) = (2k+1)^2 = 4k^2 + 4k + 1$

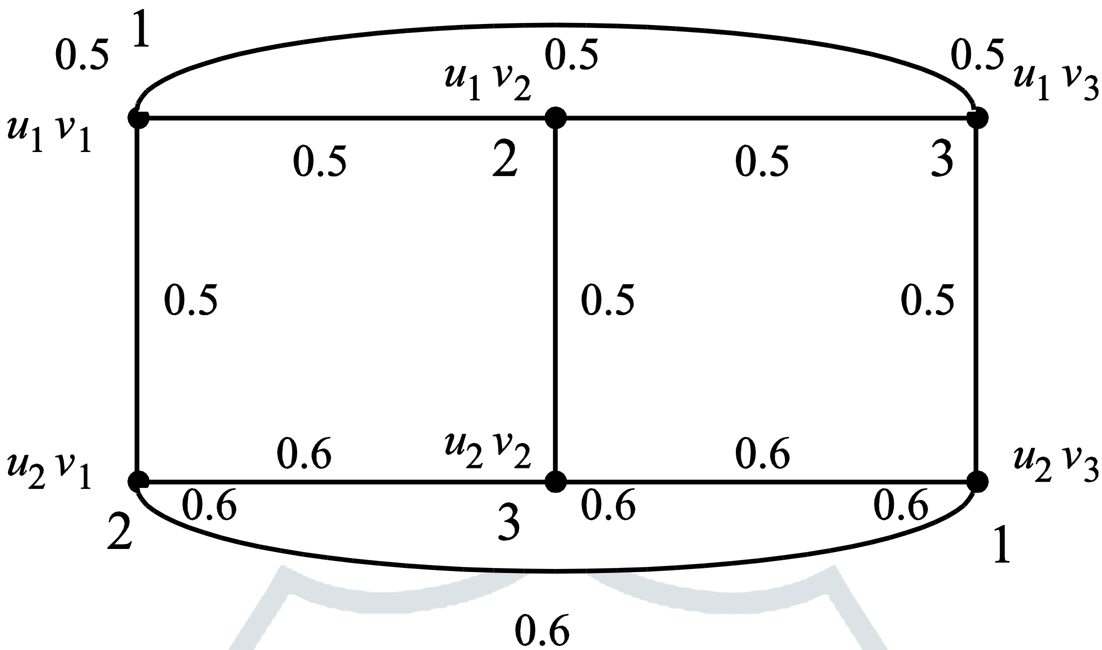
Let us colour the vertices of the graph as follows.

Let $S_1 = \{u_1v_1, \ldots, u_1v_3, \ldots, u_{2k}v_{2k}\}$ be the subset of $G(V \times H)$ and by definition of proper colouring all the vertices of S_1 are assigned with colour 1 since no two of it is adjacent.Let $S_2 = \{u_1v_3, u_2v_4, \ldots, u_{2k+1}v_{2k-1}\}$ be the subset of $V(G \times H)$ and all the vertices of S_2 are assigned with colour 2 since no two of it is adjacent.The remaining vertices of $V(G \times H)$ is $S_3 = \{u_1v_3, \ldots, u_{2k}v_{2k}\}$ assigned with colour 3.Therefore we required 3 colour for fuzzy vertex proper colouring.Hence the fuzzy chromatic number of $(G \times H)$ is $\chi^f(G \times H) = 3$.Then from the above four cases, we have $\chi^f(G \times H) = 3$ if *m* is even, *n* is odd and *m*, *n* are odd.Since we have $\chi^f(G \times H) = 2$ if *m* is odd, *n* is even and *m*, *n* are even.Hence proved.

Example:

Case (i)





$G \times H$

Cartesian product of Fuzzy path and Cycle on 2 & 3 $(P_2 \times C_3)$ with fuzzy colouring.

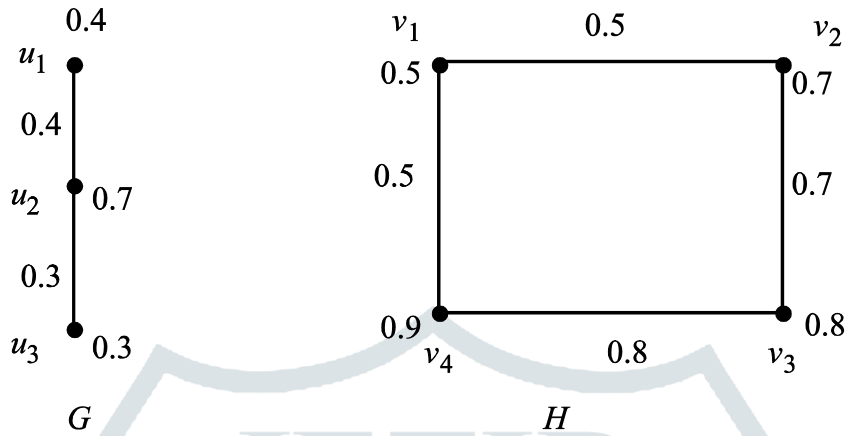
Satisfying the definition of fuzzy colouring for fuzzy graph $G \times H$ is shown in the table below.

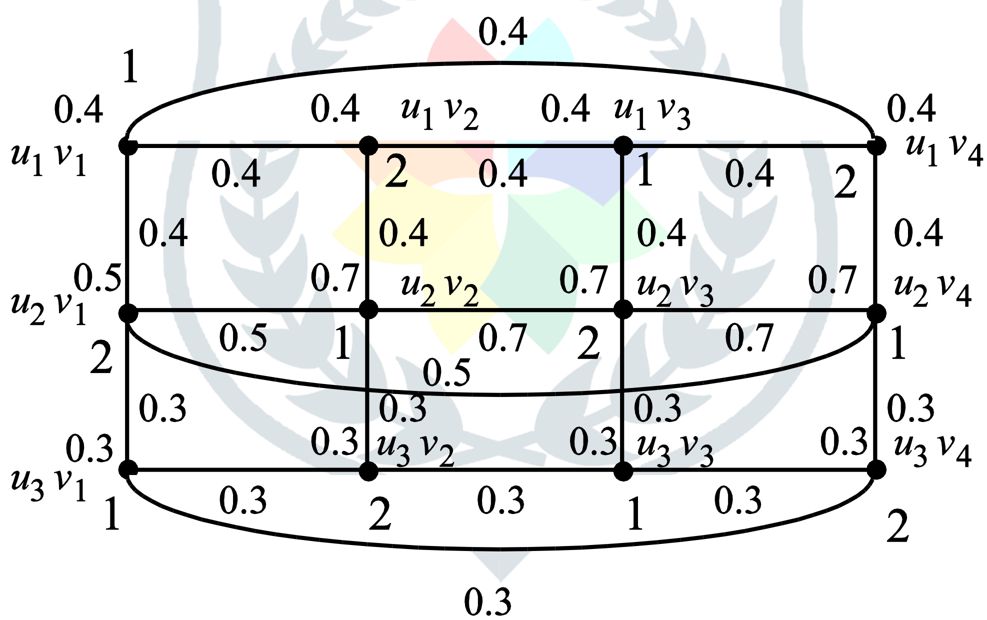
Vertices	Colour 1	Colour 2	Colour 3	$\wedge\Gamma = \sigma$ (Max γ_i)	$\gamma_i \wedge \gamma_j \wedge \gamma_k = 0$ $i, j, k = 1, 2, 3$
u_1v_1	0.5	0	0	0.5	0
u_1v_2	0	0.5	0	0.5	0
u_1v_3	0	0	0.5	0.5	0
u_2v_1	0	0.6	0	0.6	0
u_2v_2	0	0	0.6	0.6	0
u_2v_3	0.6	0	0	0	0

Fuzzy Colouringof $(P_2 \times C_3)$.

Example:

Case (ii)





$G \times H$

Cartesian product of Fuzzy path and Cycle on 3 & 4 $(P_3 \times C_4)$ with fuzzy colouring.

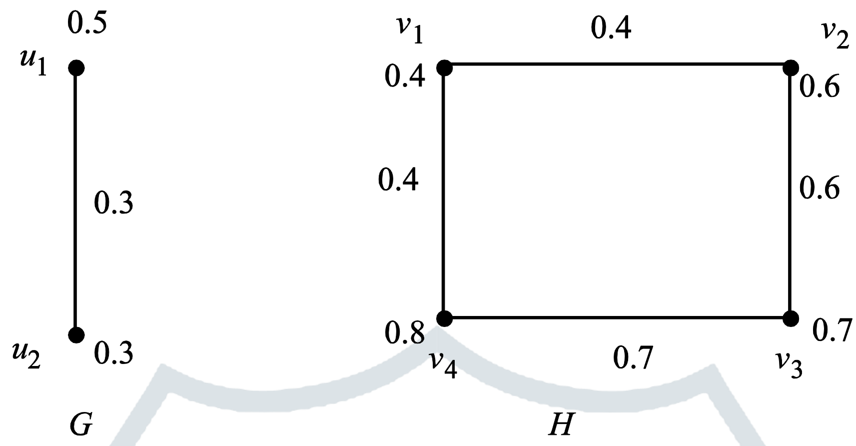
Satisfying the definition of fuzzy colouring for fuzzy graph $G \times H$ is shown in the table below.

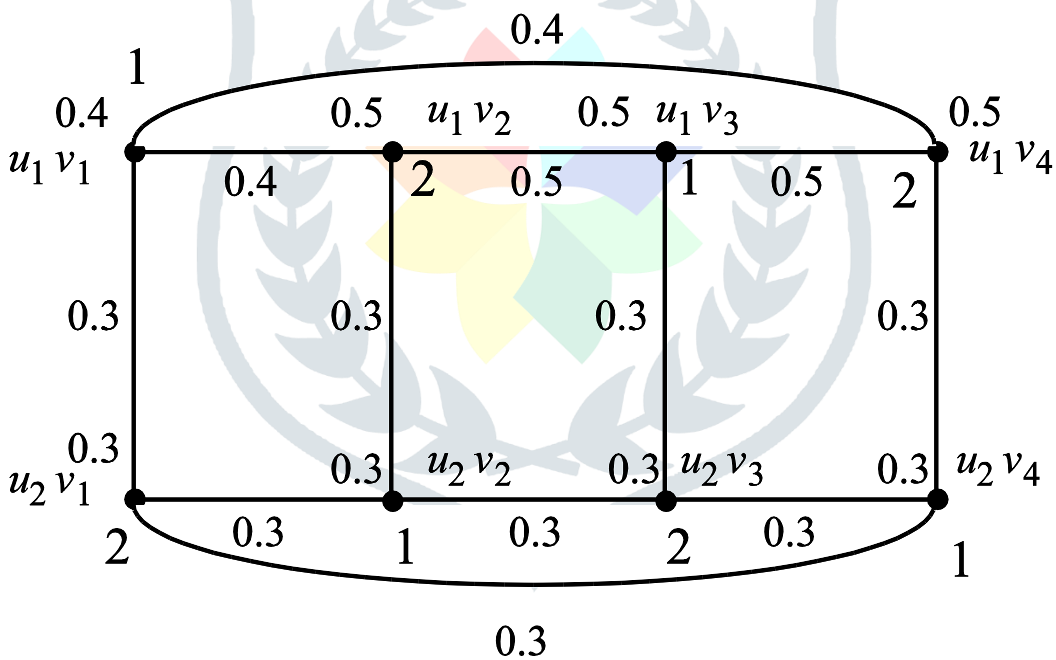
Vertices	Colour 1	Colour 2	$\wedge\Gamma = \sigma$ (Max γ_i)	$\gamma_i \wedge \gamma_j = 0$ $i, j = 1, 2$
u_1v_1	0.4	0	0.4	0
u_1v_2	0	0.4	0.4	0
u_1v_3	0.4	0	0.4	0
u_1v_4	0	0.4	0.4	0
u_2v_1	0	0.4	0.4	0
u_2v_2	0.7	0	0.7	0
u_2v_3	0	0.7	0.7	0
u_2v_4	0.7	0	0.7	0
u_3v_1	0.3	0	0.3	0
u_3v_2	0	0.3	0.3	0
u_3v_3	0.3	0	0.3	0
u_3v_4	0	0.3	0.3	0

Fuzzy Colouringof $\left(P_3 \times C_4\right)$.

Example:

Case (iii)





$G \times H$

Cartesian product of Fuzzy path and Cycle on 2 & 4 $\left(P_2 \times C_4\right)$ with fuzzy colouring.

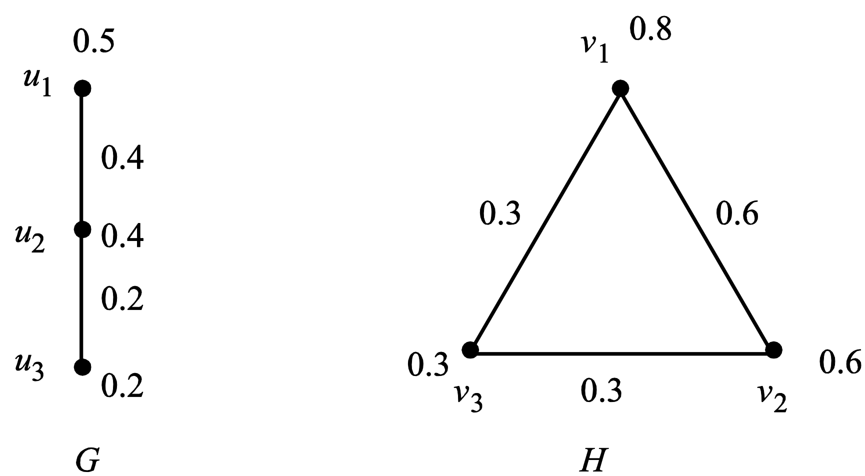
Satisfying the definition of fuzzy colouring for fuzzy graph $G \times H$ is shown in the table below.

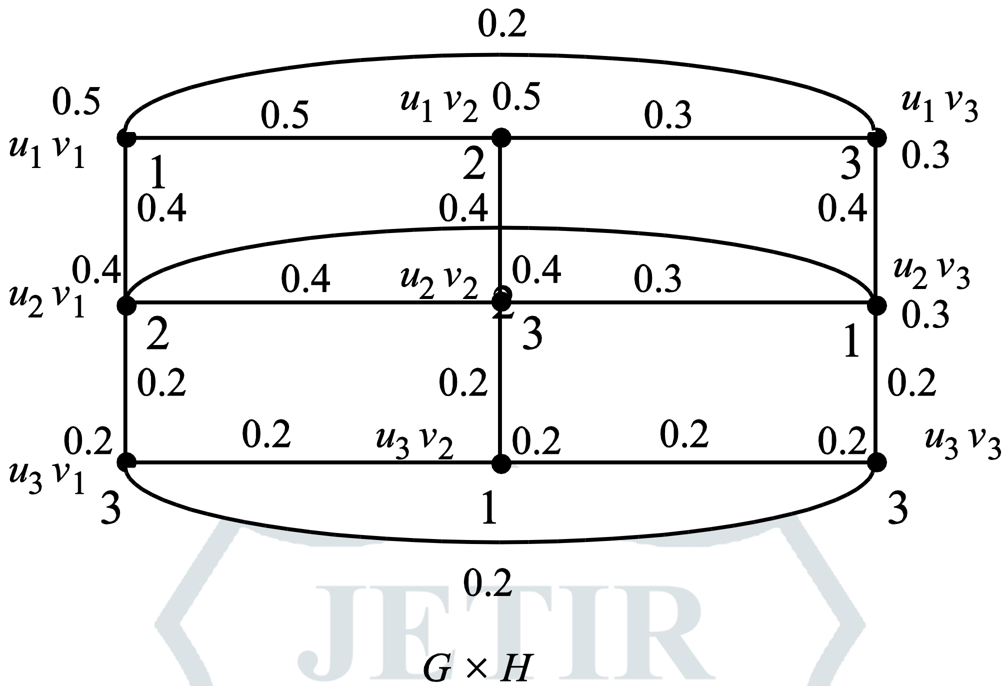
Vertices	Colour 1	Colour 2	$\wedge\Gamma = \sigma$ (Max γ_i)	$\gamma_i \wedge \gamma_j = 0$ $i, j = 1, 2$
u_1v_1	0.4	0	0.4	0
u_1v_2	0	0.5	0.5	0
u_1v_3	0.5	0	0.5	0
u_1v_4	0	0.5	0.5	0
u_2v_1	0	0.3	0.3	0
u_2v_2	0.3	0	0.3	0
u_2v_3	0	0.3	0.3	0
u_2v_4	0.3	0	0.3	0

Fuzzy Colouringof $(P_2 \times C_4)$**.**

Example:

Case (iv)





Cartesian product of Fuzzy path and Cycle on 3 & 3 $(P_3 \times C_3)$ with fuzzy colouring.

Satisfying the definition of fuzzy colouring for fuzzy graph $G \times H$ is shown in thetable below

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Vertices	Colour 1	Colour 2	Colour 3	$\wedge\Gamma = \sigma$ (Max γ_i)	$\gamma_i \wedge \gamma_j \wedge \gamma_k = 0$ $i, j, k = 1, 2, 3$
u_1v_1	0.5	0	0	0.5	0
u_1v_2	0	0.5	0	0.5	0
u_1v_3	0	0	0.3	0.3	0
u_2v_1	0	0.4	0	0.4	0
u_2v_2	0	0	0.4	0.4	0
u_2v_3	0.3	0	0	0.3	0

u_3v_1	0	0	0.2	0	0
u_3v_2	0	0.2	0	0.2	0
u_3v_3	0	0	0.2	0.2	0

Fuzzy Colouring of $(P_3 \times C_3)$.

Observation3.1.1:

If $G \times H$ is a Cartesian product of fuzzy path on *m* and fuzzy cycle on *n* vertices then $\chi^f(G \times H)$ = 3, where *m*, *n* are odd .

Observation3.1.2:

If $G \times H$ is a Cartesian product of fuzzy path on *m* and fuzzy cycle on *n* vertices then $\chi^f(G \times H)$ = 2, where *m*, *n* are even and *m*, *n* are odd and even vertices of *G* and *H*.

4.CONCLUSION

In this paper,we have established the bounds for fuzzy chromatic number Cartesian product on path and cycle of simple graph and we observed that the fuzzy chromatic number and chromatic number are same.

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