ALGEBRAIC STRUCTURES ON SR – MAGIC SQUARE OF ORDER FOUR

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ABSTRACT :

In this paper a generic definition for SR - Magic Squares is given. and group structure of SR - magic squares is discussed and SR –magic square are proved to be an abelian group. A function on SR - magic squares is also defined and it is proved to be a group homomorphism and isomorphism. The paper also deals with linear transformation, the formation of a vector space for the set of all SR - magic squares and the kernel of the mapping. And also the SR - magic squares are proved to have a ring structure and some particular SR - magic squares form commutative ring with unity. The paper also sheds light on the field structure of SR - magic squares.

Keywords: Magic Square, Magic Constant, SR - Magic Square, Abelian group, Homomorphism, Isomorphism, Kernel, Linear transformation, Vector space, Ring, Field.

1. INTRODUCTION :

While magic squares are recreational on one hand they can be treated somewhat more seriously in higher mathematics on the other hand. A normal magic square is a square array of consecutive numbers from 1 … n^2 where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, SR - magic square of order 4 have some more property. There are many recreational aspects of SR - magic squares. But, apart from the usual recreational aspects, it is found that these SR - magic squares possess advanced mathematical properties.

2. NOTATIONS AND MATHEMATICAL PRELIMINARIES

2.1 Magic Square

A magic square of order n is an n^th^ order matrix $[a_{i,j}]$ such that

$\sum_{j=1}^{n} a_{i,j} = \alpha$ for i=1, 2, 3, 4, ………n --------------- (1)

$\sum_{j=1}^{n} a_{j,i} = \alpha$ for i=1, 2, 3, 4, ………n --------------- (2)

$\sum_{i=1}^{n} (a_{i,i}) = \alpha$ --------------- (3)

$\sum_{i=1}^{n} (a_{i,n+1-i}) = \alpha$ --------------- (4)

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and equation (4) represents the co-diagonal sum and symbol $\boldsymbol{\alpha}$ represents the magic constant.

2.2 Magic Constant

The constant α in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\alpha\ [A]$

2.3 SR MAGIC SQUARE (SRMS) GENERIC DEFINITION :

A SR - magic square over a field R is a matrix $A = [a_{i,j}]$ of order 4×4 with entries in R such that, the following conditions holds.

Sum of ($a_{11} + a_{12} + a_{13} + a_{14}$) = $\alpha[A]$ -------------------------------- (1)

Sum of ($a_{21} + a_{22} + a_{23} + a_{24}$) = $\alpha[A]$ -------------------------------- (2)

Sum of ($a_{31} + a_{32} + a_{33} + a_{34}$) = $\alpha[A]$ -------------------------------- (3)

Sum of ($a_{41} + a_{42} + a_{43} + a_{44}$) = $\alpha[A]$ -------------------------------- (4)

Sum of ($a_{11} + a_{21} + a_{31} + a_{41}$) = $\alpha[A]$ -------------------------------- (5)

Sum of ($a_{12} + a_{22} + a_{32} + a_{42}$) = $\alpha[A]$ -------------------------------- (6)

Sum of ($a_{13} + a_{23} + a_{33} + a_{43}$) = $\alpha[A]$ -------------------------------- (7)

Sum of ($a_{14} + a_{24} + a_{34} + a_{44}$) = $\alpha[A]$ -------------------------------- (8)

Sum of ($a_{11} + a_{22} + a_{33} + a_{44}$) = $\alpha[A]$ -------------------------------- (9)

Sum of ($a_{14} + a_{23} + a_{32} + a_{41}$) = $\alpha[A]$ -------------------------------- (10)

Sum of ($a_{11} + a_{21} + a_{33} + a_{43}$) = $\alpha[A]$ -------------------------------- (11)

Sum of ($a_{12} + a_{22} + a_{34} + a_{44}$) = $\alpha[A]$ -------------------------------- (12)

Sum of ($a_{13} + a_{23} + a_{31} + a_{41}$) = $\alpha[A]$ -------------------------------- (13)

Sum of ($a_{14} + a_{24} + a_{32} + a_{42}$) = $\alpha[A]$ -------------------------------- (14)

Sum of ($a_{13} + a_{14} + a_{31} + a_{32}$) = $\alpha[A]$ -------------------------------- (15)

Sum of ($a_{23} + a_{24} + a_{41} + a_{42}$) = $\alpha[A]$ -------------------------------- (16)

Sum of ($a_{11} + a_{12} + a_{33} + a_{34}$) = $\alpha[A]$ -------------------------------- (17)

Sum of ($a_{21} + a_{22} + a_{43} + a_{44}$) = $\alpha[A]$ -------------------------------- (18)

Sum of ($a_{11} + a_{14} + a_{21} + a_{24}$) = $\alpha[A]$ -------------------------------- (19)

Sum of ($a_{21} + a_{24} + a_{31} + a_{34}$) = $\alpha[A]$ -------------------------------- (20)

Sum of ($a_{31} + a_{34} + a_{41} + a_{44}$) = $\alpha[A]$ -------------------------------- (21)

Sum of ($a_{11} + a_{12} + a_{41} + a_{42}$) = $\alpha[A]$ -------------------------------- (22)

Sum of ($a_{12} + a_{13} + a_{42} + a_{43}$) = $\alpha[A]$ -------------------------------- (23)

Sum of ($a_{13} + a_{14} + a_{43} + a_{44}$) = $\alpha[A]$ -------------------------------- (24)

Sum of ($a_{11} + a_{12} + a_{21} + a_{22}$) = $\alpha[A]$ -------------------------------- (25)

Sum of ($a_{12} + a_{13} + a_{22} + a_{23}$) = $\alpha[A]$ --------------------------------- (26)

Sum of ($a_{13} + a_{14} + a_{23} + a_{24}$) = $\alpha[A]$ --------------------------------- (27)

Sum of ($a_{21} + a_{22} + a_{31} + a_{32}$) = $\alpha[A]$ --------------------------------- (28)

Sum of ($a_{22} + a_{23} + a_{32} + a_{33}$) = $\alpha[A]$ --------------------------------- (29)

Sum of ($a_{23} + a_{24} + a_{33} + a_{34}$) = $\alpha[A]$ --------------------------------- (30)

Sum of ($a_{31} + a_{32} + a_{41} + a_{42}$) = $\alpha[A]$ --------------------------------- (31)

Sum of ($a_{32} + a_{33} + a_{42} + a_{43}$) = $\alpha[A]$ --------------------------------- (32)

Sum of ($a_{33} + a_{34} + a_{43} + a_{44}$) = $\alpha[A]$ --------------------------------- (33)

Sum of ($a_{11} + a_{13} + a_{31} + a_{33}$) = $\alpha[A]$ --------------------------------- (34)

Sum of ($a_{12} + a_{14} + a_{32} + a_{34}$) = $\alpha[A]$ --------------------------------- (35)

Sum of ($a_{21} + a_{23} + a_{41} + a_{43}$) = $\alpha[A]$ --------------------------------- (36)

Sum of ($a_{22} + a_{24} + a_{42} + a_{44}$) = $\alpha[A]$ --------------------------------- (37)

Sum of ($a_{11} + a_{14} + a_{41} + a_{44}$) = $\alpha[A]$ --------------------------------- (38)

Sum of ($a_{13} + a_{24} + a_{31} + a_{42}$) = $\alpha[A]$ --------------------------------- (39)

Sum of ($a_{12} + a_{21} + a_{34} + a_{43}$) = $\alpha[A]$ --------------------------------- (40)

Equation (1), (2), (3), (4) represents the row sum, equation (5), (6), (7), (8) represents the column sum, equation (9) represents the diagonal sum and equation (10) represents the co-diagonal sum.

2.3.1 Example

15	10	3	6
1	8	13	12
14	11	2	7
4	5	16	9

SR – Magic square of order 4, with magic constant $\alpha = 34$

2.4 Magic Constant (SRMS)

Given A = $[a_{i,j}]$ be a SR - magic square of order 4. Then its magic constant or magic number is defined as $\alpha[A] = \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j}$

2.5 Group

A group (G, *) is a non empty set G closed under a binary operation * such that the following axioms are satisfied.

(i) * is associative in G. i.e, a * (b * c) = (a * b) * c, ∀ a, b, c ∈ G

(ii) ∃ e ∈ G, such that e * a = a * e, ∀ a ∈ G, where e is the identity element for *.

(iii) Corresponding to each a ∈ G; ∃ b ∈ G such that a * b = b * a = e, where b is the inverse of a.

2.6 Abelian Group

A group G is abelian if its binary operation * is commutative.

2.7 Vector Space

A non-empty set V together with two operations + and . called addition and scalar multiplication respectively, is called a vector space or linear space over a field F if the following conditions are satisfied.

(i) $< V, + >$ is an abelian group.

(ii) $\forall\, \lambda \in F$ and a $\in V$, $\lambda . a \in V$.

(iii) $\forall\, \lambda \in F$ and a, b $\in V$, $\lambda(a + b) = \lambda. a + \lambda. b$

(iv) $\forall\, \lambda, \mu \in F$ and a $\in V$, $(\lambda + \mu). a = \lambda. a + \mu. b$

(v) $\forall\, \lambda, \mu \in F$ and a $\in V$, $(\lambda\, \mu). a = \lambda. (\mu\, a)$

(vi) 1 . a = a, $\forall$ a $\in V$ and 1 is the unity element of the field F.

2.8 Group homomorphism

A mapping f from a group $< G, * >$ into a group $< G', *' >$ is a homomorphism of G into G' if f (A * B) = f (A) $*'$ f (B) for all A, B $\in G$.

2.9 A one to one and onto mapping

A function : X ⟶ Y is one to one if $f\,(x_1) = f\,(x_2)$ only when $x_1 = x_2$.

The function f is onto of Y if the range of f is Y

2.10 Group isomorphism

A one to one onto homomorphism f from a group $< G, * >$ into a group $< G', *' >$ is defined as isomorphism.

2.11 Linear Transformation

Let U and V be two vector spaces over the same field F. Then a mapping $f: U \rightarrow V$ is called linear transformation of U into V if (λa+ μb) = (a) + (b) $\forall$ λ, μ $\in F$ and a, b $\in$ U.

2.12 Kernel of a Homomorphism

If φ is a homomorphism of a group G into G', then the kernel of φ is denoted as $k$er φ and is defined as $k$er φ = { g $\in$ G; φ $(g) = e'$ } , where e' is the identity of G'.

2.13 Rings

A non-empty set R together with two binary operations + and · called addition and multiplication respectively is called a ring denoted as <R, +, ·> if the following axioms are satisfied.

i. <R, +> is an abelian group.

ii. Multiplication is associative., i.e., a .($b.c$)=($a.b$).c ∀ a,b,c ∈ R

iii. Multiplication is distributive with respect to the addition,

i.e., $a.(b+c)=a.b + a.c$ (Left distributive law) and $(b+c).a=b.a + c.a$ (Right distributive law)

2.14 Commutative Ring

A ring in which the multiplication is commutative is called a commutative ring. A ring with a multiplicative identity element 1 is called a ring with unity.

2.15 Field

A ring R with at least two elements is called a field if it

i. is commutative

ii. has unity

iii. is such that each non zero element possesses multiplicative inverse.

2.16 Other Notations

1. G denotes the set of all SR-Magic Square of order 4 (SRMS).
2. G_a denotes the set of all SR - Magic Square of order 4 such that $a_{i\,j}$ = a, for every i, j = 1, 2, 3, 4. ie. If A = $[a_{i\,j}]$ ∈ G_a then each $a_{i\,j}$ = a, for some a ϵ R. we denote A = [a].
3. G_0 denotes the set of all SR - Magic Square of order 4 such that $a_{i\,j}$ = 0, for every i, j = 1, 2, 3, 4.
4. If A = $[a_{i\,j}]$ ∈ G, then its magic constant of A, $\alpha[A] = \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4} a_{i,j}$
5. If A = $[a_{i\,j}]$ ∈ G_a , then its magic constant of A, $\alpha[A]$ = 4a
6. If A = $[a_{i\,j}]$ ∈ G_0 , then its magic constant of A, $\alpha[A]$ = 0

3. PROPOSITIONS AND THEOREMS

Proposition 3.1 : If A and B are two SR – Magic Square of Order 4 with $\alpha[A]$ = a and $\alpha[B]$ = b, then C = $(\lambda + \mu)$ (A + B) is also a SR – Magic Square of Order 4 with magic constant $(\lambda + \mu)(\alpha[A] + \alpha[B])$, for every $\lambda, \mu \in R$.

Proof:

Let A = $[a_{ij}]$ and B = $[b_{ij}]$

then C = $(\lambda + \mu)$ (A + B) = [$(\lambda + \mu)$ (A + B)]

Sum of the i^{th} row elements of

$$C = \sum_{j=1}^{4} c_{ij} = (\lambda + \mu)\left(\sum_{j=1}^{4} a_{ij} + \sum_{j=1}^{4} b_{ij}\right)$$

$$= (\lambda + \mu)(a + b)$$

$= (\lambda + \mu)(\alpha[a] + \alpha[b])$

A similar computation holds for column sum, diagonal and co-diagonal sum.

From the above propositions the following results can be obtained by putting suitable values for λ and μ.

Results: For every $A, B \in G$ and $\lambda, \mu \in R$, then

3.1 a: $\lambda(A + B) \in G$, with magic constant $\alpha[\lambda(A + B)] = \lambda(\alpha[A] + \alpha[B])$

Proof: Taking $\mu = 0$ In the above proposition 3. 1

3.1 b: $(A + B) \in$ G, with magic constant $\alpha[(A + B)] = \alpha[A] + \alpha[B]$

Proof: By putting $\lambda = 1$ in result 3.1 a

3.1 c: $\lambda A \in G$, with magic constant $\alpha[\lambda A] = \lambda . \alpha[A]$

Proof: It can be easily verified by putting B=0 in result 3.1 a

3.1 d: $(\lambda + \mu)(A) \in G$, with magic constant $\alpha[(\lambda + \mu)(A)] = (\lambda + \mu)\alpha[A]$

Proof: In the Proposition 3.1 put $B = 0$, where $\in$ G.

3.1 e: $\lambda A + \mu B \in G$, with magic constant $\alpha[(\lambda A + \mu B)] = \lambda . \alpha[A] + \mu . \alpha[B]$

Proof: It can be deduced from result 3.1 b and 3.1 c.

3.1 f: $-A \in G$, with magic constant $\alpha[(-A)] = -\alpha[A]$

Proof: By Putting $\lambda = -1$ in result 3.1 c, it can be obtained.

3.1 g: $(A - B) \in$ G, with magic constant $\alpha[(A - B)] = \alpha[A] - \alpha[B]$

Proof: From the above result 3.1.b and 3.1.f it can be deduced

Theorem 3.2 : < G,+> forms an abelian group.

Proof:

Closure property :

If A, $B \in$ G , then $A + B \in$ G. (from above result 3.1 b)

Associativity :

If A, B, $C \in$ G , then $A + (B + C) = (A + B) + C \in$ G (Since matrix addition is associative.)

Existence of Identity:

There exists 0 matrix in G so that $A + 0 = 0 + A =$, where 0 acts as the identity element.

Existence of additive inverse:

For every $A \in$ G, there exists $-A \in$ G so that $A + (-A) = 0$ where $0 \in$ G (from result 3.1 f).

Commutativity:

If A, $B \in$ G , then $A + B = B + A \in$ G (Since matrix addition is commutative.)

This completes the proof.

Proposition 3.3 : G_a forms a subgroup of the abelian group G.

Proof:

It is clear that $G_a \subset$ G.

For $A, B \in G_a$; $A = [a]$ and $B = [b]$, then clearly $A - B = [a - b] \in G_a$

Thus G_a forms a subgroup of the abelian group G.

Proposition 3.4 : G_0 forms a subgroup of the abelian group G.

Proof:

It is clear that $G_0 \subset$ G

Take $A, B \in G_0$, then $\alpha[A] = 0 = \alpha[B]$

Now $\alpha[A - B] = \alpha[A] - \alpha[B] = 0$

Therefore $A - B \in G_0$.

Thus G_0 forms a subgroup of the abelian group G.

Proposition 3.5 : For all A, $B \in$ G, λ, $\mu \in R$;

a) λ(A + B) = λA + λB
b) $(\lambda + \mu).\ A = \lambda.\ A + \mu.\ A$
c) $(\lambda.\ \mu).\ A = \lambda.\ (\mu.\ A)$
d) $1.\ A = A$

Proof:

Since A, B $\in$ G; A = $[a_{ij}]$ and B = $[b_{ij}]$

a) A + B = $[a_{ij}] + [b_{ij}]$

(A + B) = λ ($[a_{ij}] + [b_{ij}]$)

= $\lambda\ [a_{ij}] + \lambda\ [b_{ij}]$

= λ A + λ B

b) $(\lambda + \mu).\ A = (\lambda + \mu).\ [a_{ij}]$

= $\lambda\ .\ [a_{ij}] + \mu.[a_{ij}]$

= λ . A+ . A

c) $(\lambda.\ \mu).\ A = (\lambda.\ \mu).\ [a_{ij}]$

= $[\lambda.\ \mu.\ (a_{ij})]$

= $\lambda.\ [\mu.\ a_{ij}]$

$= \lambda. (\mu. A)$

d) $1. A$ $= 1. [a_{ij}]$

$= [1.a_{ij}]$

$= [a_{ij}]$

$= A$

Theorem 3.6 : $< G, +, . >$ forms a vector space over the field of real numbers.

Proof:

It is an immediate consequence of Theorem 3.2 and Proposition 3.5

Theorem 3.7 : $< G_a, +, . >$ forms a vector space over the field of real numbers.

Proof:

Since $G_a \subset G$; and G is a vector space over the field of real numbers R with respect to the addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication, it is enough to show that G_a is a subspace of G.

This can be verified by the fact; for every $\lambda, \mu \epsilon R$, and $A, B \in G_a$; $\lambda A + \mu B \in G_a$

Since $A, B \in G_a$, $A = [a]$ and $B = [b]$

$\lambda A + \mu B$ $= [a] + \mu [b]$

$= [\lambda A] + [\mu B]$

$= [\lambda A + \mu B] \in G_a$

Theorem 3.8 : $< G_0, +, . >$ forms a vector space over the field of real numbers.

Proof:

Proceeding as in Proposition 3.7 it is enough to show that for every $\lambda , \mu \epsilon R$, and $A, B \in G_0$; $\lambda A + \mu B \in G_0$

Since $A, B \in G_0$; $\alpha[A] = 0$ and $\alpha[B] = 0$

Now $\alpha[\lambda A + \mu B]$

$= \lambda\, \alpha[A] + \mu\, \alpha[A]$ (From result 3.1 e)

$= \lambda . 0 + . 0 = 0$

Thus $\lambda A + \mu B \in G_0$

Proposition 3.9 : The mapping $f : G \rightarrow R$ defined by $f (A) = \alpha[A]$ for all $A \in G$ is a group homomorphism.

Proof :

Let A, B ∈ G, then

$f (A + B) = \alpha[A + B]$

$= \alpha[A]+ \alpha[B]$ (By result 3.1 b and Proposition 3.3)

Proposition 3.7 : The mapping $f : G_a \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_a$ is a group homomorphism.

Proof:

It can be easily verified since $G_a \subset$ G

Proposition 3.8 : The mapping $f : G_0 \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_0$ is a group homomorphism.

Proof:

It can be easily verified since $G_0 \subset$ G

Proposition 3.9 : The mapping $f : G_a \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_a$ is a group isomorphism.

Proof:

Let A, B $\in G_a$, A = $[a_{ij}]$, B = $[b_{ij}]$ then $\alpha[A] = 4a$ and $\alpha[B] = 4b$

(i) To show that f is one to one

f (A) = f (B)

$\Longrightarrow \alpha[A] = \alpha[B]$

$\Longrightarrow$ 4a = 4b

$\Longrightarrow$ a = b

(ii) To show that f is onto

For every a $\in R$, there exists A = $\left[\frac{a}{4}\right] \in G_a$ such that $\alpha[A] =$ a

Since f is 1−1 and onto and from Proposition 3, it can be deduced.

Proposition 3.10 : The mapping f : G $\to$ R defined by f (A) = $\alpha[A]$ for all A $\in$ G is a linear transformation.

Proof :

Let A, B $\in$ G, then

f (λA + μB) = $\alpha[\lambda$A + μB]

= $\lambda\, \alpha[A] + \mu\, \alpha[A]$ (By Result 1.4 and Theorem 3.6)

= $\lambda\, f(A) + \mu\, f(B)$

Proposition 3.11 : The mapping $f : G_a \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_a$ is a linear transformation.

Proof :

Let A, B $\in G_a$, then A = [a], B = [b] such that $\alpha[A]$ = 4a and $\alpha[B]$ = 4b

From Result 1.4 and Theorem 3.7

f (λA + μB) = $\alpha[\lambda$A + μB]

= $\lambda\, \alpha[A]\ + \mu\, \alpha[A]$

= $\lambda\, f(A)\ + \mu\, f(B)$

Hence G_a is a linear transformation.

Proposition 3.12 : The mapping $f : G_0 \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_0$ is a linear transformation.

Proof :

Let A, B $\in G_0$, then $\alpha[A]$ = 0 and $\alpha[B]$ = 0

f (λA + μB) = $\alpha[\lambda$A + μB]

= $\lambda\, \alpha[A]\ + \mu\, \alpha[A]$ (By Result 1.4 and Theorem 3.8)

= $\lambda\, f(A)\ + \mu\, f(B)$

Hence G_0 is a linear transformation.

Theorem 3.16 : The mapping $f : G_a \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_a$ is a ring homomorphism.

Proof :

Let A, B $\in G_a$, A = $[a_{ij}]$, B = $[b_{ij}]$ then $\alpha[A] = 4a$ and $\alpha[B] = 4b$

f (A + B) = α (A + B)

= $\alpha\ (A) + \ \alpha\ (B\ )$

= f (A) + f (B)

Now AB = [4ab] with $\alpha[AB]$ = 16ab

f (AB) = $\alpha\ [AB]$

$\alpha\ [AB]$ = 16ab

= 4a.4b

= α[a] . α [b]

= f (A) . f (B)

Theorem 3.17 : The mapping $f : G_a \to$ R defined by f (A) = $\alpha[A]$ for all A $\in G_a$ is a ring isomorphism.

Proof:

From Proposition 3.6 and Proposition 3.5 it can be deduced

Theorem 3.17 : < G_a, +, . > forms a ring.

Proof:

Since

a) G_a is an abelian group under matrix addition.

b) Matrix multiplication is associative and distributive over addition.

c) G_a is closed under matrix multiplication.

$< G_a, +, \; . >$ forms a ring.

Theorem 3.18 : Let $A = [a]$, $B = [b] \in G_a$, then $A.B = B.A$

Proof:

Since $A, \in G_a$

$$A = [a] = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \text{ and } B = [b] = \begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix}$$

$$\text{Then } A.B = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix} = \begin{bmatrix} 4ab & 4ab & 4ab \\ 4ab & 4ab & 4ab \\ 4ab & 4ab & 4ab \end{bmatrix}$$

$$= [4ab]$$

$$= [4ba]$$

$$= \begin{bmatrix} 4ba & 4ba & 4ba \\ 4ba & 4ba & 4ba \\ 4ba & 4ba & 4ba \end{bmatrix}$$

$$= \begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix} \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

$$= B \, . \, A$$

Hence $A.B = B.A$

Theorem 3.18 : $< G_a, +, \; . >$ is a commutative ring with unity $I_a = \left[\frac{1}{4}\right]$

Proof :

To prove $I_a = \left[\frac{1}{4}\right]$ is the unity, it is enough to prove that $A \, . \, I_a = I_a . A = A$

For $A \in G_a$, $A = [a] = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$ and $I_a = \left[\frac{1}{4}\right] = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$, clearly $I_a \in G_a$.

$$A \, . \, I_a = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} = A$$

also

$$I_a.A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} = A$$

hence $A \,.\, I_a = I_a.A = A$ and by theorem 3.17 and 3.18, $< G_a, +, \;. >$ is a commutative ring with unity $I_a = \left[\frac{1}{4}\right]$

Theorem 3.19: If $A \in G_a$, then A has a multiplicative inverse in G_a. (Here $A \neq 0$)

Proof:

Let $A \in G_a$, then $A = [a]$. Now we have to find out an element $B \in G_a$ such that $A.B = I_a$, the identity element of G_a.

Take B = $\left[\frac{1}{16a}\right]$ then clearly $B \in G_a$ and A.B = [a]. $\left[\frac{1}{16a}\right] = \left[\frac{1}{16a}\right]$. [a] = B.A = $\left[\frac{1}{4}\right] = I_a$.

Hence A has a multiplicative inverse in G_a.

Theorem 3.20: $< G_0, +, \;. >$ forms a field.

Proof:

Since $< G_0, +, \;. >$ forms a commutative ring with unity (Theorem 3.18) and it has a multiplicative inverse (Theorem 3.19), it will form a field.

4. CONCLUSION :

The study of SR - magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding SR - magic squares namely Group structure, Abelian, Vector spaces, Group homomorphism, Group isomorphism, Vector space isomorphism, Linear transformation, Kernel of transformation are described. This will help in applying SR - magic squares in different areas. Physical application of magic squares is still a new topic that needs to be explored more. Further studies are being carried out by the authors on the scope for further research and the application of SR - Magic Squares.

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