EDGE DOMINATION IN SUBDIVISION OF BLOCK GRAPHS OF GRAPHS

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Abstract: A set of edges $F \subseteq E[S(B(G))]$ is called an edge dominating set of subdivision of block graph of G, if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number of a graph $S(B(G))$ is denoted by $\gamma'[S(B(G))]$, is the minimum cardinality of an edge dominating set of $S(B(G))$. In this paper, we obtain many bonds on $\gamma'[S(B(G))]$, in terms of vertices, edges, blocks and different parameters of G and not the members of $S(B(G))$. Further we determine its relationship with other domination parameters.

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I. INTRODUCTION

All graphs considered here are simple, finite, nontrivial, separable, undirected and connected. As usual, p, q and n denote the number of vertices, edges and blocks of a graph G respectively. For graph theoretic terminology we refer F.Harary [3]. Hedetniemi and Laskar in [5] studied connected domination and further connected domination number of a graph is studied by Sampatkumar and Walikar in [9]. As usual, the maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex v is called a cut vertex if removing it from G increases the number of components of G. For any real number x, $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the greatest integer not greater than x. A graph G is called trivial if it has no edges. If G has at least one edge then G is called a nontrivial graph. A nontrivial connected graph G with at least one cut vertex is called a separable graph, otherwise a non-separable graph.

A vertex cover in a graph G is a set of vertices that covers all edges of G. The vertex covering number $\alpha_0(G)$ is a minimum cardinality of a vertex cover in G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G. A set of vertices in a graph G is called an independent set if no two vertices in the set are adjacent. The vertex independence number $\beta_0(G)$ of a graph G is the maximum cardinality of an independent set of vertices in G. The edge independence number $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges.

A nontrivial connected graph with no cut vertex is called a block. A subdivision of an edge uv is obtained by removing an edge uv, adding a new vertex w and adding edges uw and wv. For any (p, q) graph G, a subdivision graph $S(G)$ is obtained from G by subdividing each edge of G. Here, a subdivision graph S(B(G)) is obtained from B(G) by subdividing each edge of B(G).

A set $D \subseteq V(G)$ of a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. A dominating set D is a total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set in G. This concept was introduced by Cockayne, Dawes and Hedetniemi in [2].

A set F of edges in a graph $G(V, E)$ is called an edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G. The edge domination number of a graph $S(B(G))$ is denoted by $\gamma'[S(B(G))]$, is the minimum cardinality of an edge dominating set of $S(B(G))$. Edge domination number was studied by S.L. Mitchell and Hedetniemi in [7].

A dominating set D is called connected dominating set of G if the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of a graph G is the minimum cardinality of a connected dominating set in G. For any connected graph G with $\Delta(G) < p - 1$, $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$.

In this paper, many bonds on $\gamma'[S(B(G))]$, were obtained in terms of vertices, edges, blocks and other parameters of G. Also, we obtain some results on $\gamma'[S(B(G))]$, with other domination parameters of G.

II. RESULTS

Initially we establish the relation between $\gamma'[S(B(G))]$ and number of blocks of G.

Theorem 1: For any separable graph G, $\gamma'[S(B(G))] \leq n(G)$ where $n(G)$ is the number of blocks of G. Equality holds for a star graph $K_{1,4}$.

Proof: We prove the result by induction on number of blocks n of G. If $n(G) = 2$ then $\gamma'[S(B(G))] = 1 < n(G)$. Assume the result is true for all separable graphs G with $n - 1$ blocks. That is $\gamma'[S(B(G))] \leq n(G) - 1$. Let G_1 be a connected graph with n blocks. With this n^{th} block of G, only one edge will be added in $S(B(G_1))$. Then by the definition of edge dominating set, $\gamma'[S(B(G_1))] \leq (n - 1) + 1 = n$. Hence, by induction $\gamma'[S(B(G))] \leq n(G)$.

For an equality, if G is isomorphic to $K_{1,4}$, then $\gamma'[S(B(G))] = 4 = n(G)$.

The following theorem is the relation between $\gamma'[S(B(G))]$ and edges of G.

Theorem 2: For any separable graph G, $\gamma'[S(B(G))] \leq q(G)$. Equality holds if $G \cong K_{1,4}$.

Proof: Let G be a graph with n blocks. For any separable graph G, $n(G) \leq p(G) - 1$ and $p(G) - 1 \leq q(G) \Rightarrow n(G) \leq q(G)$. From Theorem 1, $\gamma'[S(B(G))] \leq n(G) \leq q(G)$. Hence, $\gamma'[S(B(G))] \leq q(G)$.

For an equality if G is isomorphic to $K_{1,4}$, then $\gamma'[S(B(G))] = 4 = q(G)$.

Now, we establish the relation between $\gamma'[S(B(G))]$, $\gamma'(G)$ and edges of G.

Theorem 3: For any separable graph G, $\gamma'(G) + \gamma'[S(B(G))] < 2q(G)$.

Proof: This result follows from Theorem 2 and the fact that for any separable graph G, $\gamma'(G) < q(G)$.

The following theorem gives the upper bound for $\gamma'[S(B(G))]$.

Theorem 4: For any separable graph G, $\gamma'[S(B(G))] < p(G)$.

Proof: Suppose $V = \{v_1, v_2, \ldots, v_i\} \subseteq V[B(G))]$ be the set of vertices in $B(G)$ corresponding to the blocks $\{B_1, B_2, \ldots, B_i\}$ in G. Then $D \subseteq E[S(B(G))]$ forms an edge dominating set of $S(B(G))$ with $\gamma'[S(B(G))] = |D|$. Since for any separable graph G, $\gamma'[S(B(G))] = |D| < p(G)$. It follows that, $\gamma'[S(B(G))] < p(G)$.

Now we obtain the following characterization.

Theorem 5: For any nontrivial tree T with at least two edges, $\gamma'[(S(B(T))] < m(T) + s(T)$ where $m(T)$ is the number of end edges in T and $s(T)$ is the number of cut vertices in T.

Proof: Let s and s' be the number of cut vertices in T and $S(B(T))$ respectively. Consider m and m' be the number of end edges in T and $S(B(T))$ respectively. Suppose $D \subseteq E[S(B(T))]$ is an edge dominating set of $S(B(T))$ with $\gamma'[S(B(T))] = |D|$. Clearly, $\gamma'[S(B(T))] = |D| < m(T) + s(T)$. Therefore $\gamma'[S(B(T))] < m(T) + s(T)$.

We thus have a result, due to Ore [8].

Theorem A [8]: If G is a (p, q) graph with no isolated vertices, then $\gamma(G) \le \frac{p}{2}$.

In the following Theorem we obtain the relation between $\gamma'[S(B(G))], \gamma(G)$ and $p(G)$.

Theorem 6: For any separable graph G, $\gamma'[S(B(G))] + \gamma(G) < \frac{3p}{2}$.

Proof: From Theorem 4 and Theorem A, $\gamma'[S(B(G))] + \gamma(G) < p(G) + \frac{p(G)}{2} = \frac{3p}{2}$.

Hence, $\gamma'[S(B(G))] + \gamma(G) < \frac{3p}{2}$.

We have a following result due to Harary [3].

Theorem B [3, P.95]: For any nontrivial (p, q) connected graph G,
$\alpha_0(G) + \beta_0(G) = p = \alpha_1(G) + \beta_1(G)$.

The following theorem relates between $\gamma'[S(B(G))]$, $\alpha_0(G)$, $\beta_0(G)$, $\alpha_1(G)$ and $\beta_1(G)$.

Theorem 7: If G is a (p, q) graph, then
$\gamma'[S(B(G))] < \alpha_0(G) + \beta_0(G) = \alpha_1(G) + \beta_1(G)$.

Proof: From Theorem 4 and Theorem B, we get
$\gamma'[S(B(G))] < \alpha_0(G) + \beta_0(G) = \alpha_1(G) + \beta_1(G)$.

The following Theorem is due to V.R.Kulli [6].

Theorem C [6, P.19]: For any graph G, $\gamma(G) \le \beta_0(G)$.

In the following Theorem, we develop the relation between $\gamma'[S(B(G))]$, $\gamma(G)$, $\beta_0(G)$ and $n(G)$.

Theorem 8: For any connected (p, q) graph G, $\gamma'[S(B(G))] + \gamma(G) \le n(G) + \beta_0(G)$.

Proof: From Theorem 1 and Theorem C, we get

$$\gamma'[S(B(G))] + \gamma(G) \le n(G) + \beta_0(G).$$

T.W.Haynes et al. [4] establish the following result.

Theorem D [4, P.165]: For any connected graph G, $\gamma_c(G) \le 2\beta_1(G)$.

In the following Theorem, we develop the relation between $\gamma'[S(B(G))]$, $\gamma_c(G)$, $\beta_1(G)$ and $q(G)$.

Theorem 9: For any connected (p, q) graph G, $\gamma'[S(B(G))] + \gamma_c(G) \le q(G) + 2\beta_1(G)$.

Proof: The result follows From Theorem 2 and Theorem D.

The following upper bound was given by V.R.Kulli[6].

Theorem E[6, P.44]: If G is connected (p, q) graph and $\Delta(G) < p - 1$, then
$\gamma_t(G) \le p - \Delta(G)$.

We obtain the following result.

Theorem 10: If G is a connected (p, q) graph and $\Delta(G) < p - 1$,

$$\gamma'[S(B(G))] + \gamma_t(G) < 2p - \Delta(G).$$

Proof: From Theorem 4 and Theorem E, we get

$$\gamma'[S(B(G))] + \gamma_t(G) < 2p - \Delta(G).$$

The following Theorem is due to S.Arumugam et al. [1].

Theorem F[1]: For any (p, q) graph G, $\gamma'(G) \le \left\lfloor \frac{p}{2} \right\rfloor$. The equality is obtained for $G = K_p$.

Now we establish the following upper bound.

Theorem 11: For any (p, q) graph G, $\gamma'[S(B(G))] + \gamma'(G) \le q(G) + \left\lfloor \frac{p}{2} \right\rfloor$.

Proof: From Theorem 2 and Theorem F, the result follows.

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