Ruler and Compass Constructions Using Galois Theory (Geometric problems of Antiquity)

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Abstract: Construction in geometry means to draw shapes, Angel's or lines accurately. These constructions use only compass, straightedge and a pencil. Greek construction a being drawn in sand and reminded of Archimede's death as recounted by Plutarch in his life of Marcellu. There are many problems in euclidean geometry which can be studied by the application of theory of field. however Roman soilder who was rather too brusquely ordered by Archimedes to stand away from his diagram just slew him on the sport.

Keywords: P-gon, n-gon, 18-gon.

1. INTRODUCTION

Galois Theory, named after Evariste Galois (1811-1832), is a showpiece of mathematical unification and it provides a connection between Theory of Groups and the Theory of Algebraic Field Extensions. It has many applications to the Theory of Equations and Geometry. Galois Theory is the theory of all the roots of polynomials and of the symmetries among them.

A Geometric Construction is the Construction of a Line, Angle or any other Geometric Figure, by using a Ruler and a Compass only. Greeks had constructions involving the use of only ruler and compass like the division of a line into arbitrarily many segments, the bisection of angles, the drawing of parallel lines and the construction of a square equal in area to any given polygon.

However, the Greeks couldn't find constructions by ruler and compass of the following some problems: the duplication of the cube, the trisection of the angle and the squaring of the circle. Galois Theory allows formulate algebraically what it means by a ruler and compass construction.

2. RULER AND COMPASS CONSTRUCTIONS

a) Two points in the plane are given to start with. These points are considered to be constructed.

b) If two points have been constructed, we may draw the line through them, or draw a circle with center at one point and passing through the

other. Such lines and circles are then considered to be constructed.

c) The points of intersection of lines and circles which have been constructed are considered to be constructed.

3. CONSTRUCTIBLE NUMBERS

A real number is said to be Constructible if it can be located in finite steps by using a ruler and a compass only.

3.1 RESULTS ON CONSTRUCTIBLE NUMBERS

i) A real number is constructible if and only if there exist a sequence of subfields of such that, and

ii) If is constructible real number, then it is algebraic, and its degree over is a power of 2, i.e. being non-negative integer and conversely.

4. EISENSTEIN'S IRREDUCIBILITY CRITERION

Let be a polynomial in If there exist a prime number such that, then is irreducible over

4.1 REMARK

Let be a polynomial. Then is irreducible over if and only if is irreducible over, being rational number.

4.1.1 Problems of Antiquity

a. Trisecting an Angle.

b. Duplicating a Cube (Delian Problem).

c. Squaring a Circle.

a. Problem of Trisecting an Angle

Statement: There exists an Angle that cannot be trisected by using Ruler and Compass only.

Solution:

We will show that the angle cannot be trisected by ruler and compass. Now if this angle can be trisected by Ruler and Compass, then the number is constructible from.

Now

Let,

Then,

We have

Hence is irreducible over.

i.e. is irreducible over by Eisenstein's Criterion.

If Galois Extension of

Then powers of 2.

Hence is not constructible from

Hence cannot be trisected.

b. Problem of Duplicating a Cube (Delian Problem)

Statement: It is impossible to construct a cube with volume equal to twice the volume of given cube by using Ruler and Compass only.

Solution:

We take a unit cube i.e. a cube of side and volume 1 units.

Thus, we want to construct a cube whose volume is 2 units.

Let side of required cube is

Then,

Taking,

We have

Hence by Eisenstein Criterion is irreducible over.

Further powers of 2.

Hence is not constructible from. Hence, we cannot double the cube by volume.

c. Problem of Squaring a Circle

Statement: It is impossible to construct a square of area equal to area of unit circle by using Ruler and Compass only.

Solution:

Assuming there be a circle of unit radius, then area of circle is units.

If is the side of the square of whose area is equal to area of unit circle,

Then

Taking

Since is not algebraic over, hence is not algebraic over

So, for any positive integer

Hence is not constructible by Ruler and Compass.

Hence, a circle cannot be squared.

5. SIMILAR PROBLEMS

It is impossible to transform circumference of unit circle into a line segment of same length by using Ruler and Compass only.

5.1 Construction of Geometric Shapes (Construction of Polygons).

Statement:

It is impossible to construct an 18-gon (An Octadecangon or 18 gon sided Polygon) by using ruler and compass only.

Solution:

The problem is equivalent to that of constructing the angle $2\pi/18=20°$.

Since, the angle20° is not constructible by ruler and compass.

Statement:

It is impossible to construct a 7-gon by using ruler and compass only.

Solution:

The construction of 7-gon is equivalent to construct an angle of $\cos 2\pi/7$.

Which can be easily shown that $\cos 2\pi/7$ satisfies an irreducible polynomial of degree 3 (not a power of 2), hence not constructible.

Statement:

An angle cannot be trisected by ruler and compass in general.

Solution:

We show that an angle 60° cannot be trisected by ruler and compass

If possible , let angle 60° can be trisected by ruler and compass

Angle 20° is constructible

Cos 20° is constructible

2cos 20° is constructible

Let a= 2cos 20°

Then cos 60° = cos (3*20)°

$=4\cos^3 20° - 3\cos 20°$

$1\backslash 2 = 4 * \frac{a}{8} - \frac{3a}{2}$

$a^3 - 3a - 1 = 0$

now the polynomial x^3 -3x-1 is irreducible over Q

x^3-3x-1 is the minimal polynomial of a

[Q(a):Q] = 3 ≠ a power of 2

a=2cos 20° is not constructible by ruler and compass .

hence the result follows.

Statement:

it is impossible to construct a cube with volume equal to double the volume of given cube using ruler and compass only.

Solution:

we can assume the edge of the given cube to be 1 so that volume of given cube is 1

if possible let a cube of double the volume of unit cube be constructible so that volume is 2

let the edge of cube to be constructed be a

then a^3=2

a^3-2=0

but x^3-2 is an irreducible polynomial over Q

x^3-2 is the minimal polynomial of an over Q

[Q($2^{1/3}$):Q]=3≠a power of 2.

$2^{1/3}$, which is side of the required cube, is not constructible by ruler and compass.

Hence cube of volume 2 cannot be constructed using ruler and compass only

Hence a cube cannot be doubled by its volume.

General Results.

1. A regular p- gon (Polygon) is constructible by ruler and compass if and only if p is of form 2r +1, r N, p being prime.

2. A regular n-gon (Polygon with n sides) is constructible by ruler and compass if and only if n is of form 2kp1p2 … ps, k ≥ 0 and pi is of form 2r +1, r N.

3. A regular n-gon (Polygon with n sides) is constructible by ruler and compass if and only if φ (n) is a power of 2.

6. CONCLUSION:

Galois theory which elaborates upon field theory and connected it with group theory which elaborated upon field theory and connect it with group theory. Galois theory can use to solve problem stemming from classical compass and straightedge problem. An angle cannot be trisected by ruler and compass a square equal in area to the area of circle.

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