COMMON FIXED-POINT THEOREMS IN BANACH SPACE

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Abstract- This paper is based on the concept of Common Fixed-Point Theorems in Banach Space. Here we establish a common fixed point theorem for self mappings, not necessarily commuting of a closed and convex subset of a Banach space, generalizing a well known result of Gregus.

Keywords- Multi-Valued Function, Fixed Point Theorems, convex subset, Banach Spaces, Euclidean norm.

I. INTRODUCTION

In this paper we establish a common fixed point theorem for self mappings, not necessarily commuting of a closed and convex subset of a Banach space, generalizing a well known result of Gregus.

Let X be a Banach space and T be a mapping of X into itself satisfying the inequality || Tx – Ty || ≤ || x – y || for all x, y in X. T is said to be non-expensive and it is well known that the class of contraction mapping and it is properly contained in the class of all continuous mapping. Browder [6,7], Goebel [15] and Kirk [20, 21] have independently proved a fixed point theorem for non-expansive mappings defined on a closed, bounded and convex subset of a uniformly convex Banach space and in spaces with richer structure. A number of generalizations of non-expensive mapping have been discussed by many authors. The works of Dotson; Emmanuele; Goebel and Zlotkiewicz [14]; Goebel [15]; Zabreiko and Krasnoselskii [45]; Kirk [20, 21]; Massa and Roux are of special significance. A comprehensive survey concerning fixed point theorems for non-expansive and related mappings can be found in Kirk [20, 21].

On the other hand, there are mappings which satisfy conditions similar to non-expansive and which possess a unique fixed point. But such mapping cannot be viewed as generalizations of non-expansive mappings. Two such examples occur recently in Gregus and Rhoades. Motivated by a contractive condition of Hardy and Rogers in this paper we extend the result of Gregus to the case of two mappings. Let C be a closed convex subset of X. By summary, assuming $b = c$ in the contractive condition of Gregus, this author proved the following result.

Theorem 1: Let T be a mapping of C into itself satisfying the inequality

(1) || Tx – Ty || ≤a. || x – y || +b . {|| Tx – x || + || Ty – y || }

For all x, y in C, where 0<a<1, b>0 and a+2b = 1. Then T has a unique fixed point.

We now prove the following theorem.

Theorem 2: Let S and T be mappings of C into itself satisfying the inequality

(2) || Sx – Ty || ≤ a. || x – y || +b. {|| Sx – x || + || Ty – y || } +c.

{|| Sx – y || + || Ty – x || }

For all x, y in C, where 0 < a < 1, b > 0 and a+2b+2c = 1 and (1 - b). c < ab. If

(3) || Tx – x || ≤ || Sx – x ||

For all x in C, then S and T have a unique common fixed point w in C. Further, w is the unique fixed point of S and T.

Proof: Let x be an arbitrary point in C. From (2), we deduce that

|| STx – Tx || ≤ a. || Tx – x || +b. {|| STx – Tx || + || Tx – x || } +c.

{|| STx – Tx || + || Tx – x || },

which implies that

(4) || STx – Tx || ≤ $\frac{a+b+c}{1-b-c}$. || Tx – x || =|| Tx – x ||.

Similarly, we have

(5) $\| TSx - Sx \| \leq \| Sx - x \|$.

Since (4) holds for all x in C, we deduce that

$$\| STSx - STx \| \leq \| TSx - Sx \|,$$

Which implies, by (3) and (5), that

(6) $\| TTSx - TSx \| \leq \| STSx - TSx \| \leq \| Sx - x \|$.

We now define the point z by

$$z = \frac{1}{2} TSx + \frac{1}{2} TTSx.$$

Then, it follows, from (6), that

(7) $2\| TSx - z \| = 2 \| TTSx - z \| = \| TTSx - TSx \| \leq \| Sx - x \|$.

Since C is convex, z belongs to C and using (2), (5), (6) and (7), we have that

(8)
$$\begin{aligned} 2\|Sz-z\| &= \|2Sz-(TSx+TTSx)\| = \| Sz - TSx \| + \| Sz - TTSx \| \\ &\leq \| Sz - TSx \| + \| Sz - TTSx \| \\ &\leq a . \| z - Sx \| + b . \{ \| Sz - z \| + \| Sx - x \| \} \\ &+ c . \{ \| Sz - z \| + \| Sx - z \| + \| TSx - z \| \} \\ &+ a . \| z - TSx \| + b . \{ \| Sz - z \| + \| Sx - x \| \} \\ &+ c . \{ \| Sz - z \| + \| TSx - z \| + \| TTSx - z \| \} \\ &\leq a . \{\| Sx-z \| + \frac{1}{2} . \|Sx-x\|\} + 2b . \{\|Sz-z\| + \| Sx- x \|\} \\ &+ c . \{ 2\| Sz - z \| + \| Sx - z \| + \frac{3}{2} . \| Sx - x \|\}. \end{aligned}$$

On the other hand, using (2), (5) and (6), we obtain that

(9)
$$\begin{aligned} 2\|Sx-z\| &= \|2S - (TSx+TTSx) \| = \|Sx - TSx \| + \| Sx - TTSx \| \\ &\leq \| Sx - TSx \| + \| Sx - TTSx \| \\ &\leq \|Sx - x \| + a . \| x - TSx \| + b . \{\| Sx - x \| + \| Sx - x \|\} \\ &+ c . \{\|Sx-x\| + \|TTSx-TSx\| + \| TSx - Sx \| + \| Sx - x \| \} \\ &\leq \| Sx - x \| + a . \{ \| Sx - x \| + \| TSx - Sx \| \} \\ &+ (2b + 4c) . \| Sx - x \| \\ &\leq (1 + 2a + 2b + 4c) . \| Sx - x \| \\ &= (3 - 2b) . \| Sx - x \|. \end{aligned}$$

It is easily seen that (8) and (9) imply that

$$\begin{aligned} 2\| Sz - z \| &\leq a . (2 - b) . \| Sx - x \| + 2b . \{\| Sx - x \| + \| Sz - z \|\} \\ &+ c . \{2\| Sz - z \| + (3 - b) . \| Sx - x \|\}. \end{aligned}$$

Consequently we have that

(10) $\| Sz - z \| \leq \lambda . \| Sx - x \|$,

Where

$$\lambda = \frac{1}{2} \left(\frac{2a-ab+2b+3c-bc}{1-b-c}\right)$$

from the assumptions on the constants a,b and c, it follows that $0 < \lambda < 1$. We claim that $h = \inf \{\| Sx - x \| : x \in C\} = 0$, otherwise, for $0 < \varepsilon < (1 - \lambda) . h/\lambda$, there exists a point $\bar{x}$ in C such that $\| S\bar{x} - \bar{x} \| \leq h + \varepsilon$ and hence (10) implies that $h \leq \| Sz - z \| \leq \lambda . \| S\bar{x} - \bar{x} \| \leq \lambda . (h+ \varepsilon) < h$, a contradiction.

Thus h = 0 and the sets

$$H_n = \{x \in C : \{\| Sx - x \| \leq \frac{1}{n} \}$$

are non-empty for any n = 1, 2, …; of course, we have

(11) $H_1 \supseteq H_2 \supseteq \ldots\ldots \supseteq H_n \supseteq \ldots\ldots$

Let $\overline{H}_n$ be the closure of H_n. We now show that

(12) diam $\overline{H}_n \leq (3 - a)/2bn$

For any n=1, 2,……. Indeed, we obtain on using (2) for all x, y in H_n,

$$\|x - y\| \leq \|Sx - x\| + \|Sx - y\|$$
$$\leq \|Sx - x\| + \|Ty - y\| + \|Sx - Ty\|$$
$$\leq \frac{2}{n} + a \,.\, \|x - y\| + b \,.\, \{\|Sx - x\| + \|Ty - y\|\}$$
$$+ c \,.\, \{\|Sx - x\| + \|x - y\| + \|Ty - y\| + \|x - y\|\}$$
$$\leq \frac{2}{n} + (a + 2c) \,.\, \|x - y\| + (2b + 2c)/n$$
$$= (3 - a)/n + (1 - 2b). \|x - y\|$$

Since (3) implies that $\|Ty - y\| \leq \|Sy - y\| \leq \frac{1}{n}$. The above inequality implies (12) since diam H_n = diam $\overline{H}_n$ and clearly it follows from (11) that

$$\overline{H}_1 \supseteq \overline{H}_2 \supseteq \ldots\ldots \supseteq \overline{H}_n \supseteq \ldots\ldots$$

Thus $\{\overline{H}_n\}$ is a decreasing sequence of non-empty subsets of C such that the sequence {diam $\overline{H}_n$} converges to zero as $n \to \infty$ by (12). Since X is complete, so is C and by Cantor's intersection theorem, there exists a point w in C such that

$$w \in \bigcap_{n=1}^{\infty} \overline{H}_n.$$

This means that $\|Sw - w\| \leq \frac{1}{n}$ for any n = 1, 2, …… and so Sw = w. Using (3), we have Tw = w. Then w is a common fixed point of S and T. Let us suppose that w' is another fixed point of S. On using (2) for x = w and y = w', we have that

$$\|w' - w\| = \|Sw' - Tw\|$$
$$\leq a \,.\, \|w' - w\| + c \,.\, \{\|w' - w\| + \|w - w'\|\}$$
$$= (a + 2c) \,.\, \|w' - w\|.$$

This implies that w' = w since a + 2c = 1 – 2b < 1. Therefore w is the unique fixed point of S and similarly it is shown that w is the unique fixed point of T. This completes the proof.

Remark: By assuming S = T and c = o, theorem 2 becomes theorem 1.

By enunciating theorem 2 for some iterates of S and T, we obtain the following result.

Theorem 3: Let S and T be mapping of C into itself satisfying the inequality

$$\|S_{p_x} - T_{q_y}\| \leq a \,.\, \|x - y\| + b. \{\|S_{p_x} - x\| + \|T_{q_y} - y\|\}$$
$$+ c. \{\|S_{p_x} - y\| + \|T_{q_y} - x\|\}$$

For all x, y in C, where p and q are positive integers and a, b, c are as in theorem 2. If

$$\|T_{q_y} - x\| \leq \|S_{p_x} - x\|$$

For all x in C, than S and T have a unique common fixed point w in C. Further, w is the unique fixed point of S and T.

Proof: By theorem 2, mapping Sp and Tq of C into itself have a unique common fixed point w in C. Since Sw = SSpw = SpSw, we deduce that Sw is also a fixed point of Sp, it follows that Sw = w. Similarly, we can prove that Tw = w and therefore w is common fixed point S and T. If w' is another fixed point of S, then we have that Spw' = w' but the uniqueness of w implies w = w'. Thus w is also the fixed point of S as well as for the mapping of T.

The following example shows the stronger generality of theorem 3 over theorem 2.

Example: Let X be the Banach space of reals with Euclidean norm and C = [0,2]. We define S and T by putting Sx = 0 if $0 \leq x < 1$, Sx = $\frac{3}{5}$ if $1 \leq x \leq 2$, Tx = 0 if $0 \leq x < 2$ and $T_2 = \frac{9}{5}$. Then the condition (2) of theorem 1 does not hold, otherwise, we should have for x =1 and y = 2.

$$\frac{6}{5} = \|S_1 - T_2\| \leq a \,.\, \|2 - 1\| + b. \{\|1 - \tfrac{3}{5}\| + \|2 - \tfrac{9}{5}\|\}$$
$$+ c. \{\|\tfrac{9}{5} - 1\| + \|2 - \tfrac{3}{5}\|\}$$

$= a + \frac{3b}{5} + \frac{11c}{5}$

$= 1 - 2b - 2c + \frac{3b}{5} + \frac{11c}{5}$

Which implies $\frac{1}{5} + \frac{7b}{5} \leq \frac{c}{5}$, i.e, 1 + 7b ≤ c, a contradiction. However, the conditions of theorem 3 are trivially satisfied for $p = q = 2$ since $S^2x = T^2x = 0$ for all x in C.

We explicitly observe that the results of this paper, for $S = T$, are not comparable with the results, where, although the contradictive condition used in more general than (2), the additional assumptions on the coefficients and the uniform convexity of X neither imply nor are implied by the assumptions of theorem 2.

Further generalizations of theorem 1, under different contractive conditions, can be found in [6].

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