SOME INVESTIGATION ON NON-STATIC SPHERICALLY SYMMETRIC MODEL IN GENERAL RELATIVITY

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Abstract- In this paper we have considered some solution of Einstein's field equation for non-static spherically symmetric metric. Here we taken e^{α} *is the function of r and t with our assumption* $e^{\alpha} = \mu_1(t)\mu_2(r)$ *under different cases g (r) = 0 and g(r) = 12 r². Pressure and density have been calculated. As these Solution giving an isotropic and homogeneous distribution of matter in space have since long been known in differential geometry. Such solutions have special interest in general relativity as they afford suitable models of a universe which is assumed to consist of isotropic and homogenous matter so these are of special interest in general relativity.*

Keywords - Symmetric Metric, Differential Geometry, General Relativity, Isotropic, Homogenous Matter.

1. INTRODUCTION

Solution of Einstein's field equations in general relativity is much discoussed problem. Solution giving an isotropic and homogeneous distribution of matter in space have since long been known in differential geometry. Such solutions have special interest in general relativity as they afford suitable models of a universe which is assumed to consist of isotropic and homogenous matter. Such a model was considered by Friendmann and Lemaitre in their solutions for the expanding universe. The field of a static fluid sphere of constant density e_o was obtained by Schwarzsohild [17] in the form.

$$(1.1) \quad ds^2 = -\frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} - r^2(de^2 + \sin^2\theta\, d\theta^2) + \left\{ A - B\sqrt{1 - \frac{r^2}{R^2}} \right\}^2 dt^2$$

where A and B are constants and $R^2 = \frac{3}{8\pi\varepsilon_0}$. Narlikar [12] gave a generalization of it in the form.

$$(1.2) \quad ds^2 = -R^2\left\{ dx^2 + \sin^2 x(de^2 + \sin^2\theta\, d\phi^2) \right\} + s^2 dt^2$$

where R = R(t) and S = S(X). The interesting conclusion is that relativity permits non-static spherical distribution of matter with a static gravitational potential, the only restriction being that either the time rate of change of the radius must be small or that it must be constant. A method for treating Einstein's field equations applied to static sphere of fluid to provide solutions in terms of known analytic functions was developed by Tolman [18]. Leibovitz [9] [10] has extensively discussed the static and non-static solutions of Einstein's field equations for the spherically symmetric distributions. The significance of the wayl conformed curvature tensor in relation to distribution of spherical symmetry, has been investigated by Narlikar and Singh [13] and it was also shown by them that the only static spherically symmetric solution of physical significance which is conformal to flat space – time is Schwarzschild's internal solution.

It is well known that in a suitable co-ordinate system, the metric of any conformally flat space differs from the Cartesian metric of flat space by a scalar factor. Rao and Patel [16] obtained explicit conformally flat form of Schwarzschild interior solution given by

(1.3)

where
$$ds^2 = \left\{ \frac{r}{rR}(P + p^{-1}) \right\}^{-2} \left\{ -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + dt^2 \right\}$$

$$P = \frac{1}{r}\left\{ \sqrt{1+r^2} - 1 \right\} \exp\left[\sqrt{1+r^2} \right] \pm \pm ....$$

and
$$R^2 = \frac{3}{8\pi\rho_0}$$

Solution of Einstein's field equations for dust like matter has been obtained by Fock [4] as

(1.4)
$$ds^2 = \left\{ 1 - \frac{A}{\sqrt{\left(x_0^2 - x_1^2 - x_2^2 - x_3^2 \right)}} \right\}^4 \times \left(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \right)$$

which is the conformally flat form of space obtained by Friedmann [5], [6], A > 0 being an arbitrary constant. Vaschuck [20] has obtained the solution of Einstein's field equations for dust like matter for the metric of the form

(1.5)
$$ds^2 = H^2(x_0, x_1, x_2, x_3) \times \left(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \right)$$

Tolman [18] developed a method for tracing Einstein's field equation applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic function. A number of new solutions were thus obtained and the properties of three of them were examined in detail. These solutions were used by Oppenheimer and Volkof [14] in the study of massive neutron cores. Krori [8] obtained exact solutions for some dense massive spheres and pointed out their astrophysical implication. Krori [8] applied the solution V of tolman in the case of Stellor structures with a variable – density core having finite density and pressure at the centre of the body. The characteristics of Tolman's solution VI with reference to constant density as well as variable density cores and their physical implications have also been discussed. Mehra, Vaidya and Kushwaha [4] have obtained a general solution of the field equations for a complete sphere having a number of shells, one above the other, or different densities. Durgapal and Gehlot [1] have obtained exact internal solutions for dense messive stars in which contros pressure and density are infinitely large. Durgapal and Gehlot [2], [3] have obtained exact solutions for a massive sphere with two different density distributions. The density being minimum at the surface varies inversely as the square of the distance from the centre. The distribution has a core of constant density and radius.

Hargreaves [7] has discussed the stability of a static spherically symmetric fluid spheres, consisting of a core of ideal gas and radiation, in which the ratio of the gas pressure to the total pressure is a small constant, and an envelope consisting of an adiabatic gas. J.P. Deleon [15] has presented two new exact analytical solutions to Einstein's field equations representing static fluid spheres with enistropic pressures while Yadav and Saini [21] have obtained an exact, static spherically symmetric solution of Einstein's field equation for the perfect fluid with $p = \rho$. Some other workers also contributes in this field [22-30].

In this paper we have considered some solution of Einstein's field equation for non-static spherically symmetric metric. Pressure and density have been calculated. As these solutions afford suitable models of a universe which is assumed to consist of isotropic and homogeneous matter so these are of special interest in general relativity.

2.   SOLUTIONS OF THE FIELD EQUATION

Here we consider the non-static metric in the form given by

(2.1) $$ds^2 = e^\beta dt^2 - e^\alpha (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$$

where a, b are functions of r and t (i.e. non-static case). The two similar solutions are given by

(2.2) $$e^\alpha = \frac{16e^{\psi(t)}}{\left(4 + \dfrac{r^2}{R^2}\right)^2}$$

(2.3) $$e^\alpha = \frac{16R^2}{A^2(4 + Kr^2)^2}, \; R = R(t)$$

Clearly e^β is functions of t only. By using a simple transformation $e^\beta \, dt^2$ can be transformed as dt^2 and thus e^α has to be expressed in the form

(2.4) $$e^\alpha = \mu_1(t)\mu_2(r)$$

The usual condition of isotropy is obtained by using (2.4) in

(2.5) $$e^{\alpha/2}\left(\alpha'' - \frac{1}{2}\alpha'^2 - \frac{\alpha'}{r}\right) = g(r)$$

We get solutions (2.1) and (2.4) by suitable adjustment of constant if we choose g (r) = 0. When g (r) = 0, we get

(2.6) $$\alpha'' - \frac{1}{2}\alpha'^2 - \frac{\alpha'}{r} = 0$$

With the condition (2.6), the most general possible solutions are given above. The first solution gives the Friedman Lemaitre model of the expanding universe and the second one gives the solutions due to Tolman [19]. We can also obtain some other solution of (2.5). Let us consider one such solution given by

(2.7) $$e^{-\alpha/2} = \upsilon(c^2 t^2 - r^2)$$

Use of (2.7) in (2.5) yields

(2.8) $$\frac{d^2\upsilon}{dz^2} + \frac{g(r)}{8r^2}.\upsilon^2 = 0$$

Where $$Z = t^2 = \frac{r^2}{A^2}$$

Case (a)

If g (R) = 0 them υ is given by

(2.9) $$y'(A^2 t^2 - r^2) = cz + d$$

where c and d are integrating constants and $A^2 z - A^2 t^2 - r^2$.

The metric in this case represents Milne's model and is given by

(2.10) $$ds^2 = dt^2 - (cz + d)^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$$

Case (b)

By choosing $g(r) = 12\, r^2$ and putting in (2.8) we get

(2.11) $$\frac{d^2\upsilon}{dx^2} = \frac{-3}{2}\upsilon^2$$

Whose solutions after some readjustment is given in terms of elliptic functions as

(2.12) $$\upsilon = 1 + \sqrt{3}\,\frac{an(cz+d)-1}{an(cz+d)+1}$$

which in terms of a series can be written as

(2.13) $$\upsilon = 1 + \sqrt{3}\,\frac{\pi}{K}\Sigma\frac{\cos(2n^{-1})(cz+d)^{\pi/2k}}{\cosh(2n^-)\pi k'/2k} - k$$

$$\pi K\,\Sigma\frac{\cos(2n-1)(cz+d)\dfrac{\pi}{2k}}{\cos h(2n-1)\dfrac{\pi K'}{2k}} + k$$

where K is the modulus, K is the quarter period and K' is complementary to K. The surviving components of the energy – momentum tensor t_j^i usually expressed as the isotropic pressure P and density e are given by

(2.14) $$8\pi P = --4r^2\left\{2\upsilon.\,d^2\upsilon/dx^2 - 3[d\upsilon/dx]^2\right\} = 3/4 + \eta$$

and

(2.15) $$8\pi\varepsilon = 4r^2\{2\upsilon.d^2\upsilon/dx^2 - 3[d\upsilon/dx]^2\} = 12/dx + \frac{3}{4} = \eta$$

These can be written for case (a) and case (b) as follows :

Case (a)

(2.16) $$8\pi P = 12c^2o^2r^2 - \frac{3}{4} + \eta$$

(2.17) $$8\pi\varepsilon = -12c(cd^2t^2 + d) + \frac{3}{4} - \eta$$

These expressions satisfy the relation

(2.18) $$\delta^\alpha/r + \frac{1}{2}(P+\varepsilon)\delta^\beta/r = 0$$

which is the relativistic analogue of dependence of pressure on gravitational potential in Newtonian theory [19]

Case (b)

Here the pressure and density are expressed in terms of elliptic functions, namely.

(2.19) $$8\pi P = 16\sqrt{3}c^2r^2\,\frac{\{2gn^2(cz+d-an(oz+d))\}}{[an(cz+d)+1]^2}$$

$$+3\frac{sn^2(cz+d)[gn^2(cz+d)+an(cz+d)]}{[an(cz+d)+1]^4} + \frac{3}{4} - \eta$$

(2.20)
$$8\pi\rho = -16\sqrt{3}\,c^2 r^2 \frac{\{2gn^2(cz+d) - an(cz+d)}{[an(cz+d)+1]^2}$$

$$+\frac{\sqrt{3}\,sn^2(cz+d)(gn^2(cz+d) + an(cz+d)]}{[an(cz+d)+1]^2}$$

$$+24\sqrt{3}\,c\left\{1 + \sqrt{3}\frac{an(cz+d)-1}{an(cz+d)+1}\right\}\frac{S_n(cz+d)an(cz+d)}{[an(cz+d)+1]^2} - \frac{3}{4} + \eta$$

The relation (2.18) though not satisfied in this case renders the left hand side of it into a perfect differential of the form

$$\frac{d}{dx}\left\{\log\left(\frac{1}{\upsilon^2}d^2\upsilon/d^2\upsilon\right)\right\} - \frac{d}{dr}\left(\frac{1}{2r}\right)$$

Besides (a) and (b) the differential equation (2.8) has apparently no other solution possible, that is, g (g) = 0 and g(r) = 12 r² afford the only solutions given above. For let us take another metric of a model having spherical symmetry, namely.

(2.2.21)
$$ds^2 = -e^\eta dr^2 - r^2 de^2 - r^2 \sin^2\theta d\phi^2 + e^\beta dt^2$$

where h and b are functions of r and t only.

The relation (2.18) is satisfied and it gives the following differential equation.

(2.2.22)
$$\frac{\delta}{\delta r}\left[\frac{1}{r}e^{\beta/2} - \eta/2\ \beta/r\right] = \frac{2}{r^3}(1-e^\eta)\exp\left(\frac{\beta}{2} - \frac{\eta}{2}\right)$$

Further in addition to this, if we suppose that the condition of isometry is also satisfied, then we get the following differential equation.

(2.2.23)
$$\frac{\delta}{\delta r}\left[\frac{1}{r^2}\frac{\delta^\beta}{\delta^r} + \frac{1}{r^3}\right]e\,xp\left(\frac{\beta}{2} - \frac{\eta}{2}\right) = \frac{-2(1+e^\eta)\exp.(\beta/2 - \eta/2)}{r^4}$$

Solving (2.22) and (2.23) for n and b we get

(2.24)
$$e^\eta = \frac{1}{1 - m^2 r^4}$$

and

(2.25)
$$e^\beta = A^2 \cos^2\left\{\frac{1}{2}\sin^-(mr^2) + \alpha\right\}$$

Here the isotropic pressure and density are found to be

(2.26)
$$8\pi P = \left[\frac{1}{r^2} - m^2 r^4/r^2 m^2 r(1 - m^2 r^4)^{1/2}\tan\left\{\frac{1}{2}\sin^{-1}(mr)^2 + \alpha\right\}\right] - \frac{1}{r^2 + \eta}$$

and

(2.27)
$$8p\varepsilon = -\frac{(1 - 6m^2 r^4 + m^4 r^8)}{r^2(1 - m^2 - r^4} + \frac{1}{r^2} - \eta$$

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