UNIFORM CONVERGENCE OF ITERATIVE COMBINATIONS OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

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Abstract- In this chapter we have studied uniform convergence of iterative combination of Bernstein-Kantorovitch polynomials. Let $f \in L^p[0,1]$, p > 1. The Bernstein-Kantorovitch polynomials are defined as

$$K_n(f,x) = (n+1)\sum_{v=0}^{n} n_{C_v} x^v (1-x)^{n-v} \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} f(t)dt \tag{1}$$

$$= (n+1)\sum_{v=0}^{n} p_{n,v}(x) \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} f(t)dt$$

Again, the iterative combinations $I_{n,k}(f,x)$ of operator sequence . $\{K_n(f,x)\}_{n\geq 1}$ is defined as

$$I_{n,k}(f,x) = \sum_{m=1}^{k} (-1)^{m+1} k_{C_m} K_n^m(f,x), k \in N,$$

where

$$K_n^2(f,x) = K_n(K_n f,x), K_n^3(f,x) = K_n(K_n^2 f,x), K_n^m(f,x) = K_n(K_n^{m-1} f,x).$$

Here, we show that $I_{n,k}(f,x)$ converges to f(x) uniformly on [0,1].

Keywords :- Iterative combinations, Bernstein-Kantorovitch polynomials, Operator sequence, Uniform-converquence.

1. INTRODUCTION AND BASIC RESULTS- Lorentz[14] defined a sequence of polynomials $\{B_n(f,x)\}_{n>1}$ for $f \in [0,1]$.

by equation (1),

$$\{B_n(f,x)\} = \sum_{v=0}^{n} p_{n,v}(x)(\frac{v}{n}), where$$

$$p_{n,v}(x) = n_{C_v} x^v (1-x)^{n-v}$$

kantorovitch modified equation (1) for $f \in L^p[0,1]$ by

$$K_n(f,x) = (n+1)\sum_{v=0}^{n} p_{n,v}(x) \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} f(t)dt \tag{2}$$

It can be also written as

$$K_n(f,x) = \int_0^1 W\,(n,x,t) f(t)dt, \qquad (3)$$

Where ,

$$W(n,x,t) = (n+1)\sum_{v=0}^{n} p_{n,v}\,(x)\psi_{n,\,v}\,(t), \qquad (4)$$

$\psi_{n,v}(t)$ is characteristic function of $\left[\frac{v}{n+1}\,,\,\frac{v+1}{n+1}\right)$

$K_n(\cdot\,,x)$ is linear positive operator from $L^p[0\,,1]\ to\ C[0\,,1]$.

It follows from (2) that

$$K_n(1,x) = 1 \quad ;\ x\epsilon[0\,,1] \qquad (5)$$

$$K_n(t,x) = \frac{2nx+1}{2(n+1)} \qquad (6)$$

$$K_n(t^2,x) = \frac{3n(n-1)x^2+6nx+1}{3(n+1)^2} \qquad (7)$$

Therefore, first and second order moments are computed as

$$\mu_1(x) = K_n(t-x,x) = \frac{1-2x}{2(n+1)} \qquad (8)$$

$$\mu_2(x) = K_n((t-x)^2,x = \frac{3(n-1)\chi+1}{3(n+1)^2} \qquad (9)$$

Where x = x(1-x)

Moreover, the general moment of r^th order of Bernstein – Kantorovitch polynomial is related to moments of Bernstein polynomial (LorentZ[14]) by

$$\mu_r(x) = \frac{n+1}{(r+1)\,x(1-x)}\quad B_{n+1}((t-x)^{r+2},x) \qquad (10)$$

The iterative combinations $I_{n,k}(f,x)$ of operator sequence {r $K_n(f,x)$, x}n>1 is defined as

$$I_{n,k}(f,x) = \sum_{m=1}^{k}(-1)^{m+1}k_{C_m}\,K_n^m(f,x) \qquad (11)$$

where

$$K_n^2(f,x) = K_n(K_nf,x),$$

$$K_n^m(f,x) = K_n(K_n^{m-1}f,x).$$

2. $I_{n,k}(.,x)$ AS AN APPROXIMATION METHOD

Here, we have shown that $I_{n,k}(.,x)$ is a method of approximation for functions in $L^p(I)$.

Lemma :- 1 ***The sequence*** $\{k_n(f,.)\}_{n\geq 1}$ ***is*** L^p ***-bounded.***

Proof :- we use Holder's inequality in summation and then in integration to obtain

$$\left|(n+1)\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} f(t)dt\right| \leq \sum_{\nu=0}^{n} p_{n,\nu}(x)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|dt$$

$$\leq \left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\left(\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|dt\right)^p\right\}^{\frac{1}{p}} \times$$

$$\times\left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\right\}^{\frac{1}{q}}$$

$$\leq \left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\left(\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|^p dt\right)\left(\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)dt\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$

$$= \left\{\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}(n+1)|f(t)|^p dt\right\}^{\frac{1}{p}} .$$

We next use Fubini's theorem to interchange in

$$\int_0^1 |K_n(f,x)|^p\,dx \leq \sum_{\nu=0}^{n}\int_0^1\int_0^1 (n+1)p_{n,\nu}(x)\,|f(t)|^p\psi_{n,\nu}(t)dtdx$$

$$= \sum_{\nu=0}^{n}\int_0^1 (n+1)\,|f(t)|^p\psi_{n,\nu}(t) \times$$

$$\times\left(\int_0^1 p_{n,\nu}(x)dx\right)dt \quad = \|f\|^p_{L^p[0,1]} . \qquad (12)$$

This prove that

$$\|K_n(f,\cdot)\|_{L^p[0,1]} \leq \|f\|_{L^p[0,1]} . \qquad (13)$$

Corollary 2:- The sequences

Proof:- We use (13) repeatedly in

$$\{K_n^m(f,\cdot)\}_{n\geq 1} \text{ and } \{I_{n,k}(f,\cdot)\}_{n\geq 1} \text{ are } L^p - \text{bounded.}$$

$$\|K_n^m(f,\cdot)\|_{L^p[0,1]} = \|K_n\ (K_n^{m-1}f,\cdot)\|_{L^p[0,1]}$$

$$\leq \|(K_n^{m-1}f,\cdot)\|_{L^p[0,1]}$$

$$\leq \cdots$$

$$\leq \|f\|_{L^p[0,1]}, \qquad (14)$$

And using (14)

$$\|I_{n,k}(f,\cdot)\|_{L^p[0,1]} \leq \sum_{m=1}^{k} {}^kC_m\ \|(K_n^m f,\cdot)\|_{L^p[0,1]}$$

$$\leq 2^k \|f\|_{L^p[0,1]}\ . \qquad (15)$$

This completes the proof.

Theorem 1:- Let $f \in c[0,1]$. then $I_{n,k}(f,.)$ converges to $(f.)$ uniform on [0,1]

Proof:- It follows from continuity of on [0,1] that for a given $\varepsilon > 0$ such that

$|f(x_1) - f(x_2)| < \varepsilon$ if $|x_1 - x_2| < \delta$.

Now,

$$\|I_{n,k}(f,x) - f(x)\|_{C[I]} \leq 2^k \|K_n^m(f(t) - f(x), x)\|_{C[I]}$$

$$\leq 2^k \|K_n(f(t) - f(x), x)\|_{C[I]}\ . \qquad (16)$$

(Analogously (14) and (15))

Let $\psi(t)$ be characteristic function of set $\{t \in [0,1]; |t - x| < \delta\}$.

$$|K_n(f(t) - f(x), x)| \leq \int_0^1 W(n,x,t)\,|f(t) - f(x)|\,dt$$

$$= \int_0^1 W(n,x,t)\,\psi(t)|f(t) - f(x)|\,dt + \int_0^1 W(n,x,t)\,(1 - \psi(t)) \times |f(t) - f(x)|\,dt$$

$$< \varepsilon \int_0^1 W(n,x,t)\,dt + \frac{2\|f\|_{C[I]}}{\delta^2} \int_0^1 W(n,x,t)\,(t-x)^2 dt\,. \qquad (17)$$

This is compounded with (5) and (9) so that for every, $x \in I$

$$|K_n(f(t) - f(x), x)| < c_1\left(\varepsilon + \frac{1}{n}\right). \qquad (18)$$

This estimate in conjunction with (16) completes the proof of the theorem.

Theorem 2 : The sequence $\{I_{n,k}(f,\cdot)\}_{n\geq 1}$ converges to f in $L^p[I]$.

Proof: Let $\{f_r\}_{r\geq 1}$ be a sequence of functions in $C[I]$converging uniformly to

Then

$$\|I_{n,k}(f,x) - f(x)\|_{L^p[I]} \le \|I_{n,k}(f - f_r, x)\|_{L^p[I]}$$

$$+ \|I_{n,k}(f_r, x) - f_r(x)\|_{L^p[I]} + \|f_r(x) - f(x)\|_{L^p[I]}$$

$$\le c_1\|f - f_r\|_{L^p[I]} + \|I_{n,k}(f_r, x) - f_r(x)\|_{L^p[I]} + \|f_r(x) - f(x)\|_{L^p[I]}$$

From coro.(2)

$$\le c_2\|f - f_r\|_{C[I]} + \|I_{n,k}(f_r, x) - f_r(x)\|_{C[I]}$$

The proof now follows from theorem 1.

3. Conclusion:- Iterative combination of Bernstein Kantorovitch polynomials $I_{n,k}(f,x)$, Converges to $f(x)$ uniformly on [0,1].

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