

Almost g^* -Normal Spaces and gg^* -Closed Sets

Jitendra Kumar and Dr. B. P. Singh

Department of Mathematics

Shyamlal Sarswati Mahavidhalaya, Shikarpur-203395, U. P. (India)

J. V. Jain (P. G.) College Saharanpur- 247001, U. P. (India)

Abstract: In this paper, we introduce, the notion of gg^* -closed and rgg^* -closed sets in topological spaces and investigate some of their properties. Further, utilizing gg^* -closed and rgg^* -closed sets, we obtain characterizations and preservation theorems for g^* -normal, almost g^* -normal and mildly g^* -normal spaces.

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1. Introduction

In 1965, Njastad [8] introduced the concept of α -open sets in topological spaces. In 1970, Levine [5] initiated the study of so called generalized closed (briefly g -closed) sets in order to extend many of the most important properties of closed sets to a large family. In 1970, Singal and Arya [10] introduced the concept of almost normal spaces. Various properties of new classes of topological spaces have been studied and the relations of these new concepts with the concepts of almost regularity have also been investigated. In 1973, Singal and Singal [11] introduced the notion of mildly normal spaces in topological spaces. In 1985, Jankovic [3] introduced the concept of rc -continuous functions. In 2000, Veera Kumar, M. K. R. S. [12] introduced the concept of g^* -closed sets in topological spaces. In 2013, Balasubramanian [2] defined rg -normality, almost rg -normality, mildly rg -normality, continue the study of further properties of rg -normality and show that these three axioms are regular open hereditary. Recently, M. C. Sharma and Hamant Kumar [9] introduced the concept of $gg\zeta^*$ -closed sets due to Kokilavani [4], and introduced the concept of almost $g\zeta^*$ -normal spaces and by using $gg\zeta^*$ -closed sets and obtained a characterization and preservation theorems for almost $g\zeta^*$ -normal spaces. Further show that this property is a topological property and it is a hereditary property only with respect to closed domain subspaces.

2. Preliminaries

Throughout this paper, spaces (X, \mathfrak{T}) , (Y, σ) , and (Z, γ) (or simply X , Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X . The closure of A and interior of A are denoted by $cl(A)$ and $int(A)$ respectively. A subset A is said to be **regular open** (resp. **regular closed**) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The finite union of regular open sets is said to be **π -open**. The complement of a π -open set is said to be **π -closed**. A is said to be **α -open** [8] if $A \subset int(cl(int(A)))$. The complement of a α -open set is said to be **α -closed**[7]. The intersection of all α -closed sets containing A is called **α -closure** [1] of A , and is denoted by $\alpha-cl(A)$. The **α -interior** [1] of A , denoted by $\alpha-int(A)$, is defined as union of all α -open sets contained in A .

2.1 Definition. A subset A of a space (X, \mathfrak{T}) is said to be

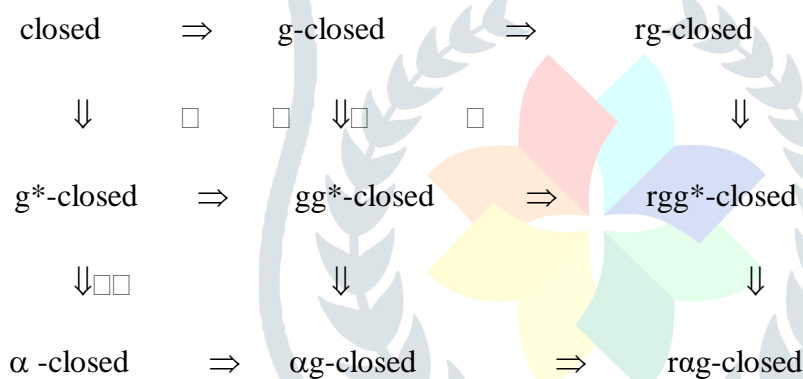
- (1) **generalized closed** (briefly **g-closed**) [5] if $cl(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.
- (2) **α -generalized closed** (briefly **α g-closed**) [6] if $\alpha-cl(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.
- (3) **rg-closed** if $cl(A) \subset U$ whenever $A \subset U$ and U is regular-open in X .
- (4) **α rg-closed** [3] if $\alpha-cl(A) \subset U$ whenever $A \subset U$ and U is regular-open in X .
- (5) **g^* -closed** [12] if $cl(A) \subset U$ whenever $A \subset U$ and U is g^* -open in X .
- (6) **g-open** (resp. **rg-open**, **α g-open**, **rag-open**, **g^***) set if the complement of A is g-closed (resp. rg-closed, α g-closed, rag-closed, g^* -closed).

The intersection of all g^* -closed sets containing A is called **g^* -closure of A**, and is denoted by **$g^*-cl(A)$** . The **g^* -interior of A**, denoted by **$g^*-int(A)$** , is defined as union of all g^* -open sets contained in A. The family of all g^* -closed (resp. g^* -open) sets of a space X is denoted by $g^*-C(X)$ (resp. $g^*-O(X)$).

2.2 Definition. A subset A of a space (X, \mathfrak{T}) is said to be

- (1) **generalized g^* -closed** (briefly **gg * -closed**) if $g^*-cl(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.
- (2) **regular generalized g^* -closed** (briefly **rgg * -closed**) if $g^*-cl(A) \subset U$ whenever $A \subset U$ and U is regular open in X .

2.3 Remark. We have the following implications for the properties of subsets:



Where none of the implications is reversible as can be seen from the following examples:

2.4 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, \{a\}, X\}$. Then $A = \{b\}$ is g-closed as well as α g-closed. Hence A is gg * -closed. But it is not closed.

2.5 Example. Let $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{\phi, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$. Then $A = \{a\}$ is α -closed as well as α g-closed. Hence A is g^* -closed. But it is not closed.

2.6 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $A = \{a, b\}$ is rg-closed as well as $rg\alpha$ -closed. Hence A is rgg * -closed. But it is not α g-closed.

2.7 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, \{a\}, X\}$. Then $A = \{a, b\}$ is g-closed as well as α g-closed. But it is not α -closed.

2.8 Example. Let $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. Then $A = \{a, b\}$ is α rg-closed as well as rgg * -closed. But it is not closed.

2.9 Theorem. Union of two rgg * -closed sets is rgg * -closed.

Proof. Let A and B be any two rgg * -closed sets. Let $A \cup B \subset U$ and U is regular open. Hence, $A \subset U, B \subset$

U. Also $g^*-cl(A) \subset U$, $g^*-cl(B) \subset U$. $g^*-cl(A \cup B) = g^*-cl(A) \cup g^*-cl(B) \subset U$. So, $A \cup B$ is rgg^* -closed.

2.10 Remark Intersection of two rgg^* -open sets is rgg^* -open.

2.11 Theorem: If $A \subset Y \subset X$ and A is rgg^* -closed in X , then A is rgg^* -closed relative to Y .

Proof: $A \subset Y \subset X$. A is rgg^* -closed in X . Let us prove that A is rgg^* -closed relative to Y . Let $A \subset Y \cap U$ where U is regular open in X . $A \subset U$ implies $g^*-cl(A) \subset U$. $Y \cap U$. So A is rgg^* -closed relative to Y .

2.12 Theorem. If A is both regular open and rgg^* -closed set in X , then A is rgg^* -closed.

Proof. $A \subset A$. A is regular open. A is rgg^* -closed. Hence $g^*-cl(A) \subset A$. But $A \subset g^*-cl(A)$. So $A = g^*-cl(A)$. Hence A is g^* -closed.

2.13. Theorem. For $x \in X$, $X - \{x\}$ is rgg^* -closed for regular open.

Proof. Let $X - \{x\}$ be not regular open. X is the only regular open set containing $X - \{x\}$. So, $g^*-cl(X - \{x\}) \subset X$. Hence $X - \{x\}$ is rgg^* -closed.

2.14 Theorem. Let A be a rgg^* -closed set if and only if $g^*-cl(A) - A$ no non empty regular closed set.

Proof. Let F be a regular closed set such that $F \subset g^*-cl(A) - A$. Then, $A \subset X - F$. A is rgg^* -closed and $X - F$ is regular open. So, $g^*-cl(A) \subset X - F$. Hence $F \subset X - g^*-cl(A) - A$. $F \subset (X - g^*-cl(A)) \cap (g^*-cl(A) - A)$. This implies $F = \phi$. Let $g^*-cl(A) - A$ contain no nonempty regular closed set. Let $A \subset U$ and U is regular open. Let $g^*-cl(A) \not\subset U$. So $g^*-cl(A) \cap U^c$ is non empty, regular closed set of $g^*-cl(A) - A$, a contradiction. Hence $g^*-cl(A) \subset U$. So A is rgg^* -closed.

2.15 Theorem. If A is rgg^* -closed and $A \subset B \subset g^*-cl(A)$, then B is rgg^* -closed.

Proof: Let $B \subset U$, where U is regular open. Then $A \subset B$ implies $A \subset U$. Since A is rgg^* -closed. $g^*-cl(A) \subset U$. $B \subset g^*-cl(A)$. Hence $g^*-cl(B) \subset g^*-cl(A) \subset U$. So, B is rgg^* -closed.

2.16 Remark. $g^*-cl(X - A) = X - g^*-int(A)$.

2.17 Theorem. $A \subset X$ is rgg^* -open if and only if $F \subset g^*-int(A)$ whenever F is regular closed and $F \subset A$.

Proof: Let A be rgg^* -open. Let F be regular closed and $F \subset A$. Then $X - A \subset X - F$. $X - F$ is regular open. $X - A$ is rgg^* -closed. $g^*-cl(X - A) \subset X - F$. $X - g^*-int(A) \subset X - F$. So, $F \subset g^*-int(A)$. Let F be regular closed and $F \subset A$ imply $F \subset g^*-int(A)$. Let $X - A \subset U$, where U is regular open. $X - U \subset A$, where $X - U$ is regular closed. Hence $X - U \subset g^*-int(A)$. So, $X - g^*-int(A) \subset U$. That is $g^*-cl(X - A) \subset U$. Hence $X - A$ is rgg^* -closed. This implies A is rgg^* -open.

2.18 Theorem. If $g^*-int(A) \subset B \subset A$ and A is rgg^* -open, B is rgg^* -open.

Proof: $X - A \subset X - B \subset X - g^*-int(A)$. That is, $X - A \subset X - B \subset g^*-cl(X - A)$. $X - A$ is rgg^* -closed. By **Theorem 2.15**, $X - B$ is rgg^* -closed B is rgg^* -open.

2.19 Remark. $g^*-int(g^*-cl(A) - A) = \phi$.

2.20 Theorem. If $A \subset X$ is rgg^* -closed, $g^*-cl(A) - A$ is rgg^* -open.

Proof. A is rgg^* -closed. Let F be a regular closed set such that $F \subset g^*-cl(A) - A$. Then by **Theorem 2.14**, $F = \phi$. So $F \subset g^*-int(g^*-cl(A) - A)$. By **Theorem 2.17**, $g^*-cl(A) - A$ is rgg^* -open.

3. Almost g^* -normal

3.1 Definition. A topological space X is said to be **g^* -normal** (resp. **rg-normal [2]**) if for every pair of disjoint closed subsets H, K of X , there exist disjoint g^* -open (resp. rg-open) sets U, V of X such that $H \subset U$ and $K \subset V$.

3.2 Definition. A topological space X is said to be **almost g^* -normal** (resp. **almost normal** [10], **almost rg-normal** [2]) if for any two disjoint closed subsets A and B of X , one of which is regular closed, there exist disjoint g^* -open (resp. open, rg-open) sets U and V of X such that $A \subset U$ and $B \subset V$.

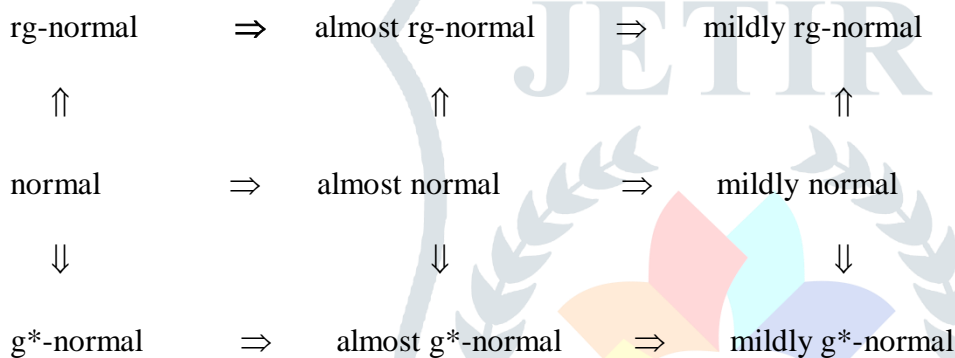
3.3 Definition. A topological space X is said to be **mildly g^* -normal** (resp. **mildly normal** [11], **mildly rg-normal** [2]) if for every pair of disjoint regular closed subsets H, K , there exist disjoint g^* -open (resp. open, rg-open) sets U, V of X such that $H \subset U$ and $K \subset V$.

3.4 Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. The pair of disjoint closed subsets of X are $A = \{a\}$ and $B = \{b, c\}$. Also $U = \{a\}$ and $V = \{b, c\}$ are open sets such that $A \subset U$ and $B \subset V$. Hence X is normal as well as almost normal.

3.5 Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then X is rg-normal.

3.6 Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then X is almost normal but it is not normal.

By the definitions and examples stated above, we have the following diagram:



Where none of the implications is reversible as can be seen from the above examples:

3.7 Theorem. For a topological space X , the following are equivalent:

- X is almost g^* -normal.
- For every closed set A and every regularly open set B containing A , there is a g^* -open set U such that $A \subset U \subset g^*\text{-cl}(U) \subset B$.
- For every regularly closed set A and every open set B containing A , there is a g^* -open set U such that $A \subset U \subset g^*\text{-cl}(U) \subset B$.
- For every pair consisting of disjoint sets A and B , one of which is closed and other is regularly closed, there exist g^* -open sets U and V such that $A \subset U, B \subset V$ and $g^*\text{-cl}(U) \cap g^*\text{-cl}(V) = \phi$.

Proof. (a) \Rightarrow (b). Let A be a closed set and let B be a regularly open set containing A . Thus $A \cap (X - B) = \phi$, where A is closed and $X - B$ is regularly closed. Therefore, there exist g^* -open sets U and V such that $A \subset U, X - B \subset V$ and $U \cap V = \phi$. Thus, $A \subset U \subset X - V \subset B$. Now, $X - V$ is closed. Therefore, $A \subset U \subset g^*\text{-cl}(U) \subset B$.

(b) \Rightarrow (c). Let A be regularly closed set and let B be an open set containing A . Then, $X - B \subset X - A$, whence $X - A$ is a regularly open set containing the closed set $X - B$. Therefore, there is a g^* -open set M such that $X - B \subset M \subset g^*\text{-cl}(M) \subset X - A$. Thus, $A \subset X - g^*\text{-cl}(M) \subset X - M \subset B$. Let $X - g^*\text{-cl}(M) = U$. Then U is g^* -open and $A \subset U \subset g^*\text{-cl}(U) \subset B$.

(c) \Rightarrow (d). Let A be a regularly closed set and B be a closed set such that $A \cap B = \phi$. Then, $A \subset X - B$ which is open. Therefore, there exists a g^* -open set M such that $A \subset M \subset g^*\text{-cl}(M) \subset X - B$. Again, M is a g^* -open set containing the regularly closed set A . Therefore, there is a g^* -open set U such that $A \subset U \subset g^*\text{-cl}(U) \subset M$. Let $X - g^*\text{-cl}(M) = V$. Then, $A \subset U, B \subset V$ and $g^*\text{-cl}(U) \cap g^*\text{-cl}(V) = \phi$.

(d) \Rightarrow (a) is obvious.

3.8 Theorem. For a topological space X , the following are equivalent:

- X is almost g^* -normal.
- For every closed set A and every regularly closed set B , there exist disjoint gg^* -open sets U and V such that $A \subset U$ and $B \subset V$.
- For every closed set A and every regularly closed set B , there exist disjoint rgg^* -open sets U and V such that $A \subset U$ and $B \subset V$.
- For every closed set A and every regularly open set B containing A , there exists a gg^* -open set U of X such that $A \subset U \subset g^*\text{-cl}(U) \subset B$.
- For every closed set A and every regularly open set B containing A , there exists a rgg^* -open set U of X such that $A \subset U \subset g^*\text{-cl}(U) \subset B$.
- For every pair of disjoint sets A and B , one of which is closed and other is regularly closed, there exist g^* -open sets U and V such that $A \subset U$ and $U \cap V = \phi$.

Proof: (a) \Rightarrow (b), (b) \Rightarrow (c), (d) \Rightarrow (e), (c) \Rightarrow (d), (e) \Rightarrow (f) and (f) \Rightarrow (a).

(a) \Rightarrow (b). Let X be an almost g^* -normal. Let A be a closed and B be a regularly closed sets in X , by assumption, there exist disjoint g^* -open sets U and V such that $A \subset U$ and $B \subset V$. Since every g^* -open is gg^* -open set, U, V are gg^* -open sets such that $A \subset U$ and $B \subset V$.

(b) \Rightarrow (c). Let A be a closed and B be a regularly closed sets in X . By assumption, there exist disjoint gg^* -open sets U and V such that $A \subset U$ and $B \subset V$. Since every gg^* -open set is rgg^* -open set, U, V are rgg^* -open sets such that $A \subset U$ and $B \subset V$.

(d) \Rightarrow (e). Let A be any closed set and B be any regularly open set containing A . By assumption, there exists a gg^* -open set U of X such that $A \subset U \subset g^*\text{-cl}(U) \subset B$. Since every gg^* -open set, there exists a rgg^* -open set U of X such that $A \subset U \subset g^*\text{-cl}(U) \subset B$.

(c) \Rightarrow (d). Let A be any closed set and B be a regularly open set containing A . By assumption, there exist disjoint rgg^* -open sets U and W such that $A \subset U$ and $X - B \subset W$. By **Theorem 2.17**, we get $X - B \subset g^*\text{-int}(W)$ and $g^*\text{-cl}(U) \cap g^*\text{-int}(W) = \phi$. Hence, $A \subset U \subset g^*\text{-cl}(U) \subset X - g^*\text{-int}(W) \subset B$.

(e) \Rightarrow (f). For any closed set A and any regularly open set B containing A . Then $A \subset X - B$ and $X - B$ is a regularly closed. By assumption, there exists a rgg^* -open set G of X such that $A \subset G \subset g^*\text{-cl}(G) \subset X - B$. Put $U = g^*\text{-int}(G)$, $V = X - g^*\text{-cl}(G)$. Then U and V are disjoint g^* -open sets of X such that $A \subset U$ and $B \subset V$.

(f) \Rightarrow (a) is obvious.

3.9 Definition. A function $f : X \rightarrow Y$ is called **rc-continuous** [] if for each regular closed set F in Y , $f(F)$ is regularly closed in X .

3.10 Definition. A function $f : X \rightarrow Y$ is called **M- g^* -open** (resp. **M- g^* -closed**) if $f(U) \in g^*\text{-O}(Y)$ ((resp. $f(U) \in g^*\text{-C}(Y)$ for each $U \in g^*\text{-O}(X)$ (resp. $U \in g^*\text{-C}(X)$)).

3.11 Definition. A function $f : X \rightarrow Y$ is called **almost g^* -irresolute** if for each $x \in X$ and g^* -neighbourhood V of $f(x)$, $g^*\text{-cl}(f(V))$ is a g^* -neighbourhood of x .

3.12 Definition. A topological space X is called **weakly g^* -regular** if for each point x and a regularly open set U containing $\{x\}$, there is a g^* -open set V such that $x \in V \subset g^*\text{-cl}(V) \subset U$.

3.13 Remark. One can prove that almost g^* -normality is also regularly closed hereditary.

3.14 Definition. A topological space X is called **almost g^* -regular** if for every regularly closed set F and each point $x \notin F$, there exist disjoint g^* -open sets U and V such that $x \in U$ and $F \subset V$.

3.15 Remark. Almost g^* -normality does not imply almost g^* -regular. Next, we prove the invariant of almost g^* -normality in the following.

3.16 Theorem. If $f : X \rightarrow Y$ is continuous M- g^* -open rc-continuous and almost g^* -irresolute surjection from an almost g^* -normal space X onto a space Y , then Y is almost g^* -normal.

Proof. Let A be a closed set and B be a regularly open set containing A . Then by rc-continuity of f , $f(A)$ is a closed set contained in the regularly open set $f(B)$. Since f is M- g^* -open and almost g^* -irresolute

surjection, we obtain $A \subset F(V) \subset g^*\text{-cl}(f(V)) \subset B$. Then again by **Theorem 3.8**, Y is almost g^* -normal.

3.17 Theorem. If $f : X \rightarrow Y$ is rc -continuous M - g^* -closed map from an almost g^* -normal space X onto a space Y , then Y is almost g^* -normal.

Proof. Easy to verify.

REFERENCES

1. Arockiarani and C. Janaki, $\pi g\alpha$ -closed set and Quasi α -normal spaces, Acta Ciencia Indica Vol. **XXXIII** M. no. **2**, (2007), 657-666.
2. S. Balasubramanian, C. Sandhya and M. D. S. Saikumar, On rg -separation axioms, Int. Jour. Of Modern Eng. Research, **2**(2012), no.6, 4001-4009.
3. D. S. Jankovic, A note on mappings of extremally disconnected spaces Acta Math. Hungar., **46**(1-2), (1985), 83-92.
4. V. Kokilavani, M. Myvizhi and M. Vivek Prabu, Generalized ζ^* -closed sets in topological spaces, Int. Jour. of Mathematical Archive, **4950**(2013), 274-279.
5. N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), **19**(1970), 89-96.
6. H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α -open and α -generalized closed sets, Mem. Fac. Sci. Kochi Univ. Math. **1**(1994), 51-63.
7. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α -open mappings, Acta. Math. Hungar., **41**(1983), 213-218.
8. Njastad, O., On some class of nearly open sets, Pacific. J. Math. , **15**(1965), 961-970.
9. M. C. Sharma and Hamant Kumar, Almost $g\zeta^*$ -normal spaces and $gg\zeta^*$ -closed sets, Int. Jour. of Science and Research Vol. **5**, Issue 6, (2016), 1288-1291.
10. M. K. Singal S. P. Arya, Almost normal and almost completely regular spaces, Glansik Mat., **5**(25), No. 1, (1970), 141-152.
11. M. K. Singal and A. R. Singal, Mildly normal spaces, Kyungpook Math. J. **13**(1973), 27-31.
12. M. K. R. S. Veera Kumar, g^* -closed sets in topological spaces, Mam. Fac. Sci. Kochi Univ. Math., **21**(2000), 1-19.