Differential Geometry of Manifolds and Surfaces

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Abstract

To study problems in geometry the technique known as Differential geometry is used. Through which in calculus, linear algebra and multi linear algebra are studied from theory of plane and space curves and of surfaces in the three-dimensional Euclidean space formed the basis for development of differential geometry during the 18th century and the 19th century. Since the late 19th century, differential geometry has grown into a field concerned more generally with the geometric structures on differentiable manifolds. The differential geometry of surfaces captures many of the key ideas and techniques characteristic of this field. Keywords: Curvature Manifolds, Riemannian geometry and surface of revolutions.

I. Introduction

Carl Friedrich Gaul (1777-1855)[1] is the father of differential geometry. He was (among many other things) a cartographer and many terms in modern differential geometry (chart, atlas, map, coordinate system, geodesic, etc.) reflect these origins. He was led to his Theorema Egregium by the question of whether it is possible to draw an accurate map of a portion of our planet. We can distinguish extrinsic differential geometry and intrinsic differential geometry. The former restricts attention to sub manifolds of Euclidean space while the latter studies manifolds equipped with a Riemannian metric. The extrinsic theory is more accessible because we can visualize curves and surfaces in R 3, but some topics can best be handled with the intrinsic theory. Organization of the paper is with respective sections: manifold, discussion of major branches of differential geometry, applications of differential geometry, differential geometry of curvature, differential geometry of surfaces and conclusions.

II. Manifold

In differential geometry, a differentiable manifold is a space which is locally similar to a Euclidean space [2]. In an n-dimensional Euclidean space any point can be specified by n real numbers. These are called the coordinates of the point. An n-D differentiable manifold is a generalization of n-dimensional Euclidean space. In a manifold it may only be possible to define coordinates locally. This is achieved by defining coordinate patches: subsets of the manifold which can be mapped into n-dimensional Euclidean space.

1.1 Kähler manifold

Kähler manifold is three mutually compatible structures; a complex structure, a Riemannian structure, and a symplectic structure. It finds important applications in the field of algebraic geometry where they represent generalizations of complex projective algebraic varieties via the Kodaira embedding theorem .

Definition 2.1.1: Symplectic viewpoint: A Kähler manifold is a symplectic manifold equipped with an integral almost-complex structure which is compatible with the symplectic form.

Definition 2.1.2: Complex viewpoint: A Kähler manifold is a Hermitian manifold whose associated Hermitian form is closed. The closed Hermitian form is called the Kähler metric.

Definition 2.1.3: Equivalence: Every Hermitian manifold is a complex manifold which comes naturally equipped with a Hermitian form and an integral, almost complex structure . Assuming that is closed, there is acanonical symplectic form defined as which is compatible with J, hence satisfying the first definition. On the other hand, any symplectic form compatible with an almost complex structure.

1.2 **Laplacians on Kähler manifolds**

Definition 2.2.1: Let * be the Hodge operator and then on an differential manifold X we can define the Laplacian as Where d is the exterior derivative and .Furthermore if X is

Kähler then d and are decomposed as , and we can define another Laplacians

, that satisfy . From these facts we obtain the Hodge

decomposition where is r - degree harmonic form and is {p, q}-degree harmonic form on X.

Remark 2.2.2: A differential form is harmonic if and only if each belong to the {i, j}-degree harmonic

form.

Definition2.2.3: A pseudo-Riemannian manifold (M, g) is a differentiable manifold M equipped with a nondegenerate, smooth, symmetric metric tensor g which, unlike a Riemannian metric, need not be positivedefinite, but must be non-degenerate. Such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero. The signature of a pseudo-Riemannian metric is (p, q) where both p and q are nonnegative.

Definition2.2.4: Lorentzian manifold: A Lorentzian manifold is an important special case of a pseudoRiemannian manifold in which the signature of the metric is $(1, n-1)$ (or sometimes $(n-1, 1)$, see sign convention). Such metrics are called Lorentzian metrics. They are named after the physicist Hendrik Lorentz.

III. Discussion Of Major Branches Of Differential Geometry

1.3 **Riemannian geometry**

It studies Riemannian manifolds, smooth manifolds with a Riemannian metric. This is a concept of distance expressed by means of a smooth positive definite symmetric bilinear form defined on the tangent space at each point. Riemannian geometry generalizes Euclidean geometry to spaces that are not necessarily flat, although they still resemble the Euclidean space at each point infinitesimally, i.e. in the first order of approximation. Various concepts based on length, such as the arc length of curves, area of plane regions, and volume of solids all possess natural analogues in Riemannian geometry. The notion of a directional derivative of a function from multivariable calculus is extended in Riemannian geometry to the notion of a covariant derivative of a tensor. Many concepts and techniques of analysis and differential equations have been generalized to the setting of Riemannian manifolds.A distance-preserving diffeomorphism between Riemannian manifolds is called an isometry. This notion can also be defined locally, i.e. for small neighborhoods of points. Any two regular curves are locally isometric. In higher dimensions, the Riemann curvature tensor is an important point wise invariant associated to a Riemannian manifold that measures how close it is to being flat. An important class of Riemannian manifolds is the Riemannian symmetric spaces, whose curvature is not necessarily constant. These are the closest analogues to the "ordinary" plane and space considered in Euclidean and non-Euclidean geometry.

1.4 Pseudo-Riemannian geometry

Pseudo-Riemannian geometry generalizes Riemannian geometry to the case in which the metric tensor need not be positive-definite. A special case of this is a Lorentzian manifold, which is the mathematical basis of Einstein's general relativity theory of gravity.

1.5 Finsler geometry

Finsler geometry has the Finsler manifold as the main object of study. This is a differential manifold with a Finsler metric, i.e. a Banach norm defined on each tangent space. A Finsler metric is a much more general structure than a Riemannian metric. Definition: A Finsler structure on a manifold M is a function such that: for all x, y in TM, F is infinitely differentiable in The vertical Hessian of is positive definite.

1.6 Symplectic geometry

Symplectic geometry is the study of symplectic manifolds. An almost symplectic manifold is a differentiable manifold equipped with a smoothly varying non-degenerate skew-symmetric bilinear form on each tangent space, i.e., a non degenerate 2-form ω, called the symplectic form. A symplectic manifold is an almost symplectic manifold for which the symplectic form ω is closed: . Definition: A diffeomorphism between two symplectic manifolds which preserves the symplectic form is called a symplectomorphism. Non-degenerate skew-symmetric bilinear forms can only exist on even dimensional vector spaces, so symplectic manifolds necessarily have even dimension. In dimension 2, a symplectic manifold is just a surface endowed with an area form and a symplectomorphism is an area-preserving diffeomorphism.

1.7 Complex and Kähler geometry

Definition: A real manifold , endowed with a tensor of type i.e. a vector bundle endomorphism (called an almost complex structure) It follows from this definition that an almost complex manifold is even dimensional. Definition: An almost complex manifold is called complex if , where is a tensor of type related to , called the Nijenhuis tensor (or sometimes the torsion). Remark: An almost complex manifold is complex if and only if it admits a holomorphic. Definition: An almost Hermitian structure is given by an almost complex structure J, along with a Riemannian metric g, satisfying the compatibility condition Definition: An almost Hermitian structure defines naturally a differential two-form The following two conditions are equivalent: 1. 2. where is the Levi-Civita connection of . In this case, is called a Kähler structure, and a Kähler manifold is a manifold endowed with a Kähler structure. In particular, a Kähler manifold is both a complex and a symplectic manifold. A large class of Kähler manifolds (the class of Hodge manifolds) is given by all the smooth complex projective varieties.

1.8 CR geometry

CR geometry is the study of the intrinsic geometry of boundaries of domains in complex manifolds.

1.9 Synthetic differential geometry

Synthetic differential geometry is a reformulation of differential geometry in the language of topos theory, in the context of an intuitionistic logic characterized by a rejection of the law of excluded middle. There are several insights that allow for such a reformulation. The first is that most of the analytic data for describing the class of smooth manifolds can be encoded into certain fibre bundles on manifolds: namely bundles of jets . The second insight is that the operation of assigning a bundle of jets to a smooth manifold is functorial in nature. The third insight is that over a certain category, these are representable functors. Furthermore, their representatives are related to the algebras of dual numbers, so that smooth infinitesimal analysis may be used.Synthetic differential geometry can serve as a platform for formulating certain otherwise obscure or confusing notions from differential geometry. For example, the meaning of what it means to benatural (or invariant) has a particularly simple expression, even though the formulation in classical differential geometry may be quite difficult.

1.10 Abstract differential geometry

The adjective abstract has often been applied to differential geometry before, but the abstract differential geometry (ADG) of this article is a form of differential geometry without the calculusnotion of smoothness, developed by Anastasios Mallios and others from 1998 onwards. Instead of calculus, an axiomatic treatment of differential geometry is built via sheaf theory and sheaf cohomology using vector sheaves in place of bundles based on arbitrary topological spaces. Mallios says non commutative geometry can be considered a special case of ADG, and that ADG is similar to synthetic differential geometry.

1.11 Discrete differential geometry

Discrete differential geometry is the study of discrete counterparts of notions in differential geometry. Instead of smooth curves and surfaces, there are polygons, meshes, and simplicial complexes. It is used in the study of computer graphics and topological combinatorics.

IV. Applications Of Differential Geometry

In physics:

a) Differential geometry is the language in which Einstein's general theory of relativity is expressed. According to the theory, the universe is a smooth manifold equipped with a pseudo-Riemannian metric, which describes the curvature of space-time. Understanding this curvature is essential for the positioning of satellites into orbit around the earth. Differential geometry is also indispensable in the study of gravitational lensing and black holes.

b) Differential forms are used in the study of electromagnetism.

c) Differential geometry has applications to both Lagrangian mechanics and Hamiltonian mechanics. Symplectic manifolds in particular can be used to study Hamiltonian systems.

d) Riemannian geometry and contact geometry have been used to construct the formalism of geometro thermodynamics which has found applications in classical equilibrium thermodynamics.

In economics [2]: differential geometry has applications to the field of econometrics. Geometric modeling (including computer graphics) and computer-aided geometric design draw on ideas from differential geometry. In engineering, differential geometry can be applied to solve problems in digital signal processing. In probability, statistics, and information theory, one can interpret various structures as Riemannian manifolds, which yields the field of information geometry, particularly via the Fisher information metric. In structural geology, differential geometry is used to analyze and describe geologic structures. In computer vision, differential geometry is used to analyze shapes.

In image processing [3], differential geometry is used to process and analyse data on non-flat surfaces.

In wireless communications [4], Grassmanian manifold is used for beam forming techniques in multiple antenna systems.

V. Differential Geometry Of Surfaces

Surface with various additional structures, most often, a Riemannian metric. Surfaces have been extensively studied from various perspectives: Extrinsically: Relating to their embedding in Euclidean space Intrinsically: Reflecting their properties determined solely by the distance within the surface as measured along curves on the surface. Carl Friedrich Gauss (1825-1827) showed that curvature was an intrinsic property of a surface, independent of its isometric embedding in Euclidean space. Surfaces naturally arise as graphs of functions of a pair of variables, and sometimes appear in parametric form or as loci associated to space curves. Lie groups can be used to describe surfaces of constant Gaussian curvature; they also provide an essential ingredient in the modern approach to intrinsic differential geometry through connections. This is well illustrated by the non-linear EulerLagrange equations in the calculus of variations: although Euler developed the one variable equations to understand geodesics, defined independently of an embedding, one of Lagrange's main applications of the two variable equations was to minimal surfaces, a concept that can only be defined in terms of an embedding.

VII. Conclusion

Differential geometry is normally considered as a speculation of the Riemannian geometry. The historical backdrop of improvement of Finsler geometry can be partitioned into four periods. The primary period of the historical backdrop of advancement of Finster geometry started in 1924, when three geometricians J.H. Taylor, J.L. Synge and L. Berwald at the same time began work in this field. Berwald is the main man who has presented the idea of association in the hypothesis of Finsler spaces. He is the maker of Finsler geometry and, besides, the author. He has built up a hypothesis with specific reference to the hypothesis of bend in which the

Ricci lesa does not hold great. J.H. taylor gave the name 'Finsler space' to the complex outfitted with this summed up metric. The second time frame started in 1934, when E. Cartan distributed his proposition on Finsler geometry. He demonstrated that it was to be sure conceivable to characterize association coefficients and subsequently covariant subordinates with the end goal that the Ricci lemma is fulfilled. On this premise Cartan built up the hypothesis of curvature tensor and torsion. Every single consequent examination considering the geometry of Finsler spaces were ruled by this approach. A few mathematicians, for example, E.T. Davies , Golab , H. Hombu , O. Varga , V.V. Wagner have considered Finsler g•ometry along Cartan's approach. They have communicated the conclusion that the hypothesis has achieved its last shape. This hypothesis makes certain gadgets, which fundamentally includes the thought of a space whose components are not the purposes of the basic complex, but rather the line-components of the last mentioned, which shapes a (2n-1) — dimensional assortment. This encourages what Cartan called 'Euclidean association' which by method for specific proposes might be gotten extraordinarily from the crucial metric capacity.

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